Structure and Asymptotic Expansion of Multiple Harmonic Sums

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ABSTRACT

We prove that the algebra of *multiple harmonic sums* is isomorphic to a *shuffle algebra*. So the multiple harmonic sums $\{H_{\mathbf{s}}\}$, indexed by the compositions $\mathbf{s} = (s_1, \cdots, s_r)$, are $\mathbb{R}$-linearly independent as real functions defined over $\mathbb{N}$. We deduce then the algorithm to obtain the asymptotic expansion of multiple harmonic sums.

Categories and Subject Descriptors

G.2.1 [**Discrete Mathematics**]: Combinatorics—*combinatorial algorithms, generating functions*

General Terms

Algorithms, Languages

Keywords

Polylogarithms, multiple harmonic sums, Lyndon words, polyzêtas

1. INTRODUCTION

Let $\mathbb{N}_{>0}$ be the set of positive integers. The harmonic sums

$$\mathrm{H}_s(N) = \sum_{n=1}^{N} \frac{1}{n^s}, \qquad (s \in \mathbb{N}_{>0}, N \in \mathbb{N}_{>0}) \tag{1}$$

can be generalized to any composition $\mathbf{s}$ of length $r \geq 0$, i.e. a sequence of positive integers $\mathbf{s} = (s_1, \ldots, s_r)$ by putting

$$\mathrm{H}_{\mathbf{s}}(N) = \sum_{N \geq n_1 > \cdots > n_r > 0} \frac{1}{{n_1}^{s_1} \ldots {n_r}^{s_r}} \tag{2}$$

with the convention $\mathrm{H}_{\mathbf{s}}(N) \equiv 1$ when $\mathbf{s}$ is the empty composition.

It can be proved that the limit

$$\zeta(\mathbf{s}) = \lim_{N \to \infty} \mathrm{H}_{\mathbf{s}}(N) \tag{3}$$

exists if and only if the composition $\mathbf{s}$ is empty or if $s_1 \neq 1$. In this case, we will say that $\mathbf{s}$ is a *convergent* composition. The values $\zeta(\mathbf{s})$ are called "Multiple Zeta Values" (MZV). Harmonic sums and MZV arise in high-energy particule physics [3] and in analysis of algorithms [5].

We consider the $\mathbb{R}$-vector space $\mathcal{H}_{\mathbb{R}}$ generated by the $\mathrm{H}_{\mathbf{s}}$, seen as functions from $\mathbb{N}$ to $\mathbb{R}$. The theory of quasi-symmetric functions shows that the $\{\mathrm{H}_{\mathbf{s}}(N)\}_{\mathbf{s}}$ satisfy *shuffle* relations. In particular, the product of two harmonic functions is a sum of harmonic functions : for all $a, b \in \mathbb{N}_{>0}$, we have

$$\mathrm{H}_a(N) \cdot \mathrm{H}_b(N) = \mathrm{H}_{a,b}(N) + \mathrm{H}_{b,a}(N) + \mathrm{H}_{a+b}(N). \tag{4}$$

So, the vector-space $\mathcal{H}_{\mathbb{R}}$ is closed under product. The main result of this article is to prove that in $\mathcal{H}_{\mathbb{R}}$, the functions $\{H_{\mathbf{s}}\}_{\mathbf{s}}$ are linearly independent. As a consequence, $\mathcal{H}_{\mathbb{R}}$ appears to be *isomorphic* to some shuffle algebra noted $\left(\mathbb{R}\langle Y\rangle, \, ⧢ \right)$. The structure of this algebra is well known and Hoffman showed that it is freely generated by Lyndon words on the alphabet Y.

Let $\mathcal{H}^0_{\mathbb{R}}$ be the $\mathbb{R}$-algebra generated by the functions $\mathrm{H}_{\mathbf{s}}$ when $\mathbf{s}$ describes the set of all convergent compositions. We show that

$$\mathcal{H}_{\mathbb{R}} \simeq \mathcal{H}^0_{\mathbb{R}}[\mathrm{H}_1] \tag{5}$$

i.e. that any harmonic function can be decomposed *uniquely* in a univariate polynomial, on the sums $\mathrm{H}_1(N) = \sum_{n=1}^{N} n^{-1}$.

This decomposition is obtained thanks to a variant of Taylor expansion for univariate polynomials, by defining a derivation D in the shuffle algebra $\left(\mathbb{R}\langle Y\rangle, \, ⧢ \right)$. In fact, this decomposition provides an asymptotic expansion, up to order 0, of $\mathrm{H}_{\mathbf{s}}(N)$ as $N \to \infty$. We can then deduce an asymptotic expansion, up to *any* order, by using the second form of the Euler-Mac Laurin summation formula.

Our result of linear independance of the functions $\mathrm{H}_{\mathbf{s}}$ lies on the $\mathbb{C}$-linear independance of the *polylogarithm* functions

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(z is a complex number so that $|z| < 1$)

$$\mathrm{Li}_{\mathbf{s}}(z) = \sum_{n_1 > \cdots > n_l > 0} \frac{z^{n_1}}{n_1^{s_1} \cdots n_l^{s_l}}, \tag{6}$$

first proved in [9], then resumed in [13, 14]. In this article, harmonic functions are seen as Taylor coefficients

$$\frac{1}{1-z} \mathrm{Li}_{\mathbf{s}}(z) = \sum_{N=0}^{\infty} \mathrm{H}_{\mathbf{s}}(N) z^N. \tag{7}$$

By using the combinatorics developed by M. Bigotte in [2], it is possible to consider a similar generalization for the harmonic sums related to *coloured* MZV

$$\zeta\binom{\xi_1,\ldots,\xi_l}{s_1,\ldots,s_l} = \sum_{n_1 > \cdots > n_l > 0} \frac{{\xi_1}^{n_1} \ldots {\xi_l}^{n_l}}{{n_1}^{s_1} \ldots {n_l}^{s_l}}, \tag{8}$$

where $\xi_1, \ldots, \xi_l$ are roots of unity.

2. BACKGROUND

2.1 How to shuffle?

We consider the *non* commutative alphabet $Y = \{y_n \mid n \in \mathbb{N}_{>0}\}$. As usual, the set of all words over Y is denoted Y^* and the empty word is denoted ϵ. The length of the word w is denoted $|w|$ and the word w resulting from the concatenation of two words u and v is the word $w = uv$.

Let R be a commutative ring containing $\mathbb{Q}$. A polynomial $p \in R\langle Y\rangle$ is a linear combination of words, with coefficients in R. The coefficient of the word w in polynomial p is noted $(p|w)$ and therefore

$$p = \sum_{w \in Y^*} (p|w) w, \qquad (p|w) \in R. \tag{9}$$

The concatenation product for words is extended to polynomials by linearity.

The shuffle product of two words $u = y_i u'$ and $v = y_j v'$ in Y^* is recursively defined by $\epsilon ⧢ w = w$, for all $w \in Y^*$ and

$$u ⧢ v = y_i(u' ⧢ v) + y_j(u ⧢ v') + y_{i+j}(u' ⧢ v'), \tag{10}$$

with $i, j \in \mathbb{N}_{>0}$ and $u', v' \in Y^*$.

This product is also extended to $R\langle Y\rangle$ by linearity. Provided with this shuffle product, $R\langle Y\rangle$ becomes an associative and commutative R-algebra noted $\big(R\langle Y\rangle, ⧢\big)$.

We can totally order Y by putting $y_i < y_j$ if $i > j$. So y_1 is the biggest letter of the alphabet Y. *Lexicographic* order is then recursively defined on words $w \in Y^*$ by

$$\begin{cases} \epsilon < w \text{ for } w \in Y^* \setminus \epsilon \\ y_i u < y_j v \text{ if } i > j \text{ or if } i = j \text{ and } u < v. \end{cases}$$

A nonempty word w is called a Lyndon word if it is strictly smaller (for lexicographic order) than any of its proper right factors, i.e. $w < v$ for any factorization $w = uv$ with $u \neq \epsilon$ and $v \neq \epsilon$. Let Lyndon(Y) denotes the set of Lyndon words over Y.

Let

$$\mathrm{C}^0 = R \oplus (R\langle Y\rangle \setminus y_1 R\langle Y\rangle). \tag{11}$$

Hoffman generalized Radford theorem in the following way

THEOREM 1 ([12, 10]). *One has*

$$(R\langle Y\rangle, ⧢) \simeq (R[\mathrm{Lyndon}(Y)], ⧢) = (\mathrm{C}^0[y_1], ⧢).$$

This means that every polynomial in $R\langle Y\rangle$, for shuffle product, can be decomposed uniquely in a linear combination of shuffle products of Lyndon words.

2.2 Quasi monomial functions

Let $T = \{t_n \mid n \in \mathbb{N}_{>0}\}$ a countable *commutative* set. To each word $w = y_{s_1} \ldots y_{s_l} \in Y^*$, we associate the quasi monomial function

$$M_w = \sum_{n_1 > \cdots > n_l > 0} t_{n_1}^{s_1} \cdots t_{n_l}^{s_l}, \tag{12}$$

which is a formal series on the letters $t_n \in T$, with coefficients in $\mathbb{N}$. For the empty word, we define $M_\epsilon = 1$. Thanks to Gessel works, it is known that the product of quasi monomial functions is a sum of quasi monomial functions. Moreover, we have the following identity between formal series

$$M_u M_v = M_{u ⧢ v}, \qquad (u, v \in Y^*). \tag{13}$$

2.3 Polylogarithms

Let us associate to any word $w = y_{s_1} \cdots y_{s_r}$ the polylogarithm $\mathrm{Li}_w(z)$ defined for $|z| < 1$ by

$$\mathrm{Li}_w(z) = \sum_{n_1 > \cdots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \cdots n_r^{s_r}}, \tag{14}$$

for $r > 0$ and by $\mathrm{Li}_\epsilon(z) = 1$, for the empty word ϵ.

THEOREM 2 ([9]). *The functions* Li_w*, for* $w \in Y^*$*, are* $\mathbb{C}$*-linearly independent.*

In fact, this result can be improved by replacing $\mathbb{C}$ by any algebra of analytic functions defined over $\mathbb{C} - \{0, 1\}$, for example $\mathbb{C}[z, 1/z, 1/(z-1)]$, and the proof lies on an explicit evaluation of the monodromy group of Li_w.

3. HARMONIC SUMS ALGEBRA

3.1 Harmonic sums

DEFINITION 1. *Let* $w = y_{s_1} \ldots y_{s_r} \in Y^*$*. Then, for* $N \geq r \geq 1$*, the harmonic sum* $\mathrm{H}_w(N)$ *is defined as*

$$\begin{aligned} \mathrm{H}_w(N) &= \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}} \\ \textit{and} \quad \mathrm{H}_w(N) &= 0 \quad \textit{for } 0 \leq N < r. \end{aligned}$$

If $|w| = 0$*, we put* $\mathrm{H}_\epsilon(N) = 1$*, for any* $N \geq 0$*.*

LEMMA 1. *Let* $w = y_{s_1} \ldots y_{s_r} \in Y^*$*. The sum* $\mathrm{H}_w(N)$ *is convergent, when* $N \to \infty$*, if and only if* $s_1 > 1$*.*

In this case, this limit is nothing but the polyzta (or MZV [15]) $\zeta(w)$, and thus, by an Abel's theorem

$$\lim_{N \to \infty} \mathrm{H}_w(N) = \zeta(w) = \lim_{z \to 1} \mathrm{Li}_w(z). \tag{15}$$

Then, the word w is said *convergent*. A polynomial of $R\langle Y\rangle$ is said convergent when it is a linear combination of convergent words.

LEMMA 2. *For* $w = y_s w'$*, we have*

$$\mathrm{H}_w(N) = \sum_{k=1}^{N} \frac{\mathrm{H}_{w'}(k-1)}{k^s}$$

COROLLARY 1. *For $w = y_s w'$, we have*

$$\zeta(w) = \sum_{k\geq 1} \frac{\mathrm{H}_{w'}(k-1)}{l^s}, \quad (16)$$

$$\mathrm{H}_w(N+1) - \mathrm{H}_w(N) = (N+1)^{-s}\mathrm{H}_{w'}(N) \quad (17)$$

LEMMA 3 ([10]). *For any words u and v, we have*

$$\mathrm{H}_{u \sqcup\!\sqcup v}(N) = \mathrm{H}_u(N)\,\mathrm{H}_v(N).$$

PROOF. The harmonic sum $\mathrm{H}_w(N)$ can be obtained by specialization of the quasi-monomial function M_w at $t_i = 1/i$ if $1 \leq i \leq N$ and $t_i = 0$ if $i > N$. By (13), $\mathrm{H}_w(N)$ satisfies the expected result. □

3.2 Generating series

DEFINITION 2 ([6]). *For any word $w \in Y^*$, let P_w be the ordinary generating series of $\{\mathrm{H}_w(N)\}_{N\geq 0}$:*

$$\mathrm{P}_w(z) = \sum_{N\geq 0} \mathrm{H}_w(N)z^N, \quad \textit{with} \quad \mathrm{P}_\epsilon(z) = \frac{1}{1-z}.$$

PROPOSITION 1 ([6]). *For any word $w \in Y^*$ and for any complex number z satisfying $|z| < 1$, one has*

$$\mathrm{Li}_w(z) = (1-z)\mathrm{P}_w(z).$$

PROOF. Since $\mathrm{P}_w(z) = \sum_{N\geq 0} \mathrm{H}_w(N)z^N$, it is known that the series expansion of $(1-z)\mathrm{P}_w(z)$ is given by

$$(1-z)\mathrm{P}_w(z) = \mathrm{H}_w(0) + \sum_{N\geq 1} [\mathrm{H}_w(N) - \mathrm{H}_w(N-1)]z^N.$$

But, by (17), for $w = y_{s_1}w'$,

$$\mathrm{H}_w(N) - \mathrm{H}_w(N-1) = N^{-s_1}\mathrm{H}_{w'}(N-1),$$

so

$$(1-z)\mathrm{P}_w(z) = \mathrm{H}_w(0) + \sum_{N\geq 1} \frac{\mathrm{H}_{w'}(N-1)}{N^{s_1}} z^N = \mathrm{Li}_w(z).$$

□

DEFINITION 3. *The Hadamard product $\odot$ is a bilinear function from $\mathbb{C}[[z]] \times \mathbb{C}[[z]]$ to $\mathbb{C}[[z]]$ defined, for all integers n and m,by*

$$z^n \odot z^m = \begin{cases} z^n \text{ if } n = m, \\ 0 \text{ if } n \neq m. \end{cases}$$

Thus, $\sum_{n=0}^{\infty} a_n z^n \odot \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} a_n b_n z^n$.

PROPOSITION 2. *For $u, v \in Y^*$, one has*

$$\mathrm{P}_u(z) \odot \mathrm{P}_v(z) = \mathrm{P}_{u \sqcup\!\sqcup v}(z).$$

PROOF. By Lemma 3,

$$\sum_{N\geq 0} \mathrm{H}_u(N)z^N \odot \sum_{N\geq 0} \mathrm{H}_v(N)z^N = \sum_{N\geq 0} \mathrm{H}_u(N)\mathrm{H}_v(N)z^N = \sum_{N\geq 0} \mathrm{H}_{u \sqcup\!\sqcup v}(N)z^N.$$

□

COROLLARY 2 ([6]). *Extented by linearity, the map $\mathrm{P} : u \mapsto \mathrm{P}_u$ is an isomorphism from $(R\langle Y\rangle, \sqcup\!\sqcup\,)$ to the Hadamard algebra of $\{\mathrm{P}_w\}_{w\in Y^*}$.*

PROOF. Proposition 2 gives P as an algebra morphism and by Theorem 2, P is the expected isomorphism. □

Thanks to the relations existing between $\mathrm{Li}_w(1-z)$ and $\mathrm{Li}_w(z)$ [8], we can precise the asymptotic behaviour of Li_w in the neighbourhood of 1. For example,

$$\mathrm{Li}_{2,1,1}(1-t) = -\mathrm{Li}_4(t) + \log(t)\,\mathrm{Li}_3(t) - \frac{1}{2}\log(t)^2\,\mathrm{Li}_2(t) + \frac{1}{6}\log(t)^3\,\mathrm{Li}_1(t) + \frac{2}{5}\zeta(2)^2.$$

So, we find, by formula (14), the expansion of $\mathrm{Li}_{2,1,1}(1-\varepsilon)$ and, by dividing it by ε, we find the one of $\mathrm{P}_{2,1,1}(1-\varepsilon)$:

$$\mathrm{P}_{2,1,1}(1-\varepsilon) = \frac{2}{5}\zeta(2)^2\frac{1}{\varepsilon} + \frac{1}{6}\log^3\varepsilon - \frac{1}{2}\log^2\varepsilon + \log\varepsilon - 1 + \frac{\varepsilon}{12}\log^3\varepsilon - \frac{\varepsilon}{8}\log^2\varepsilon + \frac{\varepsilon}{8}\log\varepsilon + \mathrm{O}(\varepsilon)$$

From this, we can also deduce the expansion of the Taylor coefficients of $\mathrm{P}(z)$ (see [4]).

3.3 Algebra $\mathcal{H}_{\mathbb{R}}$

DEFINITION 4. *The algebra $\mathcal{H}_{\mathbb{R}}$ of harmonic sums is defined as the $\mathbb{R}$-vector space $\mathcal{H}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}(H_w \mid w \in Y^*)$, equipped with the ordinary product.*

From Corollary 2, we deduce then

PROPOSITION 3. *The map $\mathrm{H} : u \mapsto \mathrm{H}_u$ is an isomorphism from $(R\langle Y\rangle, \sqcup\!\sqcup\,)$ to the algebra $\mathcal{H}_{\mathbb{R}}$.*

Since Lyndon(Y) generate freely the shuffle algebra then

COROLLARY 3. *Any harmonic sum in $\mathcal{H}_{\mathbb{R}}$ can be decomposed,* uniquely, *as a polynomial on the series H_l, for $l \in$* Lyndon(Y), *i.e.* $\mathcal{H}_{\mathbb{R}} \simeq \mathbb{R}[\mathrm{H}_l;\ l \in \mathrm{Lyndon}(Y)]$.

LEMMA 4. *Any $l \in$ Lyndon(Y) is convergent if and only if $l \neq y_1$. Any convergent polynomial can be decomposed uniquely as shuffle of convergent Lyndon words.*

PROOF. By definition, a Lyndon word l is strictly smaller (for lexicographic order) than any of its proper right factors. So, if $l = y_1 u$, with $u \in Y^*$, we have $y_1 u < u$ which is impossible (remind that y_1 is the *greatest* letter of Y). Thus, the only Lyndon word beginning by y_1 is y_1 itself, and our first statement is proved.

The second one is based on the remark : if y_1 appears as a factor (for $\sqcup\!\sqcup$) in the Radford decomposition of w, then this word begins by y_1. Since a convergent polynomial contains convergent terms, which do not begin by y_1, the statement is proved. □

PROPOSITION 4. *Every harmonic sum $\mathrm{H}_w \in \mathcal{H}_{\mathbb{R}}$ can be decomposed in a unique way in a univariate polynomial in H_1, with coefficients in the convergent harmonic sums. This can also be expressed as follows :*

$$\mathcal{H}_{\mathbb{R}} \simeq \mathcal{H}^0_{\mathbb{R}}[\mathrm{H}_1],$$

where $\mathcal{H}^0_{\mathbb{R}}$ is defined as the $\mathbb{R}$-algebra generated by the functions H_w, for all convergent words $w \in \mathrm{C}^0$.

EXAMPLE – The Radford decomposition gives, in Lyndon basis, $y_1y_4y_2 = y_1 \sqcup\!\sqcup y_4y_2 - y_5y_2 - y_4y_1y_2 - y_4y_2y_1 - y_4y_3$. Thus, $\mathrm{H}_{1,4,2} = \mathrm{H}_1\mathrm{H}_{4,2} - \mathrm{H}_{5,2} - \mathrm{H}_{4,1,2} - \mathrm{H}_{4,2,1} - \mathrm{H}_{4,3}$. □

By Proposition 3, we deduce $\ker \mathrm{H} = \{0\}$. In other words,

PROPOSITION 5. *The harmonic sums H_w, for $w \in Y^*$ are $\mathbb{R}$-linearly independent.*

4. ASYMPTOTIC EXPANSIONS

We are going to construct a recursive algorithm to find the asymptotic expansion of H_w. For that, considering any real sequence $\{s_n\}_{n\in\mathbb{N}}$, we will define $\mathtt{AS}_q(s_n)$ as the asymptotic expansion up to order q of s_n, i.e. so that

$$s_n - \mathtt{AS}_q(s_n) = \mathrm{O}(n^{-q}).$$

4.1 Euler Mac-Laurin formula

Let $\{B_n\}_{n\in\mathbb{N}}$ be the set of Bernoulli numbers obtained in the expansion of the following series

$$\sum_{n\geq 0} B_n \frac{z^n}{n!} = \frac{z}{\exp(z)-1} = \frac{z\exp(-z)}{1-\exp(-z)},$$

and $\{B_n(\cdot)\}_{n\in\mathbb{N}}$ the Bernoulli polynomials defined by

$$\frac{x\exp(tx)}{\exp(x)-1} = \sum_{n\geq 0} B_n(t)\frac{x^n}{n!}.$$

We need the second form of Euler-Maclaurin summation [11] given by, for all integers q, M, N with $N > M$,

$$\sum_{n=M}^{N} f(n) = \int_M^N f(x)dx + \frac{f(M)+f(N)}{2} + \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!}\left(f^{(2j-1)}(N) - f^{(2j-1)}(M)\right) + R_{2m} \quad (18)$$

where $R_m = \dfrac{1}{(2m+1)!}\displaystyle\int_M^N B_{2m+1}(x-[x]) f^{(2m+1)}(x)dx.$

Lemma 5 ([1]). *One has*

$$\mathrm{H}_1(N) = \log N + \gamma - \sum_{k=1}^{q-1} \frac{B_k}{k}\frac{1}{N^k} + \mathrm{O}\left(\frac{1}{N^q}\right)$$

$$\mathrm{H}_r(N) = \zeta(r) - \frac{1}{(r-1)N^{r-1}} - \sum_{k=r}^{q-1} \frac{B_{k-r+1}}{k-r+1}\binom{k-1}{r-1}\frac{1}{N^k} + \mathrm{O}\left(\frac{1}{N^q}\right),$$

with $r \geq 2$.

Proof. With the function $f(x) = x^{-r}$, the summation (18) between $M = 1$ and N gives the expected results. □

Lemma 6. *One has, for any integer* $q \geq 2$,

$$\begin{aligned}
&\sum_{k=2}^{N} \frac{\log(k-1)}{k^q} \\
&= K + \frac{\log N}{(1-q)N^{q-1}} - \sum_{i=1}^{\infty} \frac{1}{(1-q)iN^{q-1+i}} \\
&- \sum_{i=q-1}^{\infty} \frac{1}{(q-1)iN^i} + \frac{\log N}{2N^q} - \sum_{i=1}^{\infty} \frac{1}{2iN^{q+i}} \\
&+ \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!}(-1)^{2j} \\
&\times\Bigg[-(q)_{2j-1}\frac{\log N}{N^{q+2j-1}} + (q)_{2j-1}\sum_{i=1}^{\infty}\frac{1}{iN^{q+2j-1+i}} + \\
&\sum_{k=0}^{2j-2}\binom{2j-1}{k}(2j-2-k)!(q)_k\sum_{i=0}^{\infty}\frac{(-1)^i(2j-k-2)_i}{i!N^{2j+q-1+i}}\Bigg],
\end{aligned}$$

where $K = \sum_{k=2}^{+\infty} \log(k-1)k^{-q}$.

So we deduce the asympotic expansion up to order $q+2$, which will appear to be very useful afterwards :

$$\begin{aligned}
\sum_{k=2}^{N} \frac{\log(k-1)}{k^q} &= K + \frac{\log(N)}{(1-q)N^{q-1}} - \frac{1}{(q-1)^2N^{q-1}} \\
&+ \frac{\log(N)}{2N^q} + \frac{1}{qN^q} - \frac{qB_2}{2}\frac{\log(N)}{N^{q+1}} \\
&+ \left(\frac{B_2}{2} - \frac{q^2-4q-3}{2q^2-2}\right)\frac{1}{N^{q+1}} + \mathrm{O}\left(\frac{1}{N^{q+2}}\right)
\end{aligned}$$

with $K = \sum_{k=2}^{+\infty} \log(k-1)k^{-q}$.

Proof. Let $q > 0$ $f(x) = \log(x)(x+1)^{-q}$. We use the Euler-Maclaurin summation (18) from $M = 1$ to $N-1$, which leads us to calculate each term involved in this sum :

$$\begin{aligned}
\int_1^{N-1} \frac{\log(x)}{(x+1)^q}dx &= \frac{\log(N-1)}{(1-q)N^{q-1}} + \frac{\log(N-1)}{q-1} \\
&+ \frac{1}{1-q}\log\left(\frac{N}{2}\right) \\
&+ \frac{1}{q-1}\sum_{j=1}^{q-2}\frac{1}{j}\left(\frac{1}{N^j} - \frac{1}{2^j}\right) \\
\frac{f(1)+f(N-1)}{2} &= \frac{\log(N-1)}{2N^q}, \\
f^{(2j-1)}(x) &= \sum_{k=0}^{2j-2}\binom{2j-1}{k}\frac{(2j-2-k)!(q)_k}{x^{2j-k-1}(x+1)^{q+k}} \\
&- (q)_{2j-1}\frac{\log(x)}{(x+1)^{q+2j-1}}.
\end{aligned}$$

where $(s)_k = \Gamma(s+k)/\Gamma(s)$ for $k \in \mathbb{N}$.
We just need to insert the previous terms in the summation (18), expand $\log(N-1)$, and make $m \to \infty$. □

4.2 Taylor algorithm

By Theorem 1, any $w \in Y^*$ can be expressed as follows,

$$w = \sum_{k=0}^{|w|} c_k(w) ⧢ \frac{y_1^{⧢ k}}{k!}. \quad (19)$$

We want to calculate the convergent polynomials $c_k(w) \in \mathrm{C}^0$. For that, let $D : R\langle Y\rangle \to R\langle Y\rangle$ be the linear application defined, for each $p \in R\langle Y\rangle$ and for each word $w \in Y^*$ by the duality

$$(Dp|w) = (p|y_1 w). \quad (20)$$

In particular, $Dw = 0$ when w is convergent and $D(y_1w) = w$ for each word $w \in Y^*$. We can prove that D is a derivation for the shuffle product ⧢.

In the following sequence, all *products* and *powers* will be carried out with the shuffle product ⧢.

Proposition 6. *Let* $w \in Y^*$, *a word of length* $|w|$. *Then the polynomials* $c_k(w)$ *are given by*

$$c_k(w) = \sum_{i=0}^{|w|-k} \frac{(-y_1)^i D^i}{i!}\ D^k(w).$$

Since $D^k w = 0$ as soon as $k > |w|$, this formula can be summered as follows :

$$c_k(w) = e^{-y_1 D} D^k(w)$$

with the convention $\exp(-y_1D) = \sum_{i\geq 0}(-y_1)^i D^i/i!$, i.e. by making D and y_1 commute.

PROOF. For a polynomial $p \in \mathbb{R}[X]$ of degree l, the Taylor expansion is finite, and given by

$$p(x) = p(y) + Dp(y)(x-y) + \cdots + \frac{D^l p(y)}{l!}(x-y)^l.$$

So, taking $x=0$, $y=y_1$, $p=D^k(w)$, we find

$$c_k(w) = D^k(w) - D\,D^k(w)y_1 + \cdots + \frac{D^l}{l!}D^k(w)(-y_1)^l.$$

□

EXAMPLE – Let $w = y_1y_4y_2$. Note that $D(w) = y_4y_2$ and so that $D^k(w) = 0$, for $k \geq 2$. By using Proposition 6,

$$\begin{aligned} c_0 &= \sum_{i=0}^{3} \frac{(-y_1)^i D^i}{i!}(y_1y_4y_2) = y_1y_4y_2 - y_1 ⧢ y_4y_2 \\ &= -y_4y_1y_2 - y_4y_2y_1 - y_4y_3 - y_5y_2 \\ c_1 &= \sum_{i=0}^{2} \frac{(-y_1)^i D^i}{i!}(y_4y_2) = y_4y_2 \\ c_k &= 0 \quad \text{for } k \geq 2 \end{aligned}$$

So we get $y_1y_4y_2 = c_0 + c_1 ⧢ y_1 = -y_4y_1y_2 - y_4y_2y_1 - y_4y_3 - y_5y_2 + y_4y_2 ⧢ y_1$. Note that, in this case, Taylor algorithm gives directly the Radford decomposition. □

4.3 Algorithm for asymptotic expansions

We now are going to use both previous tools (Euler MacLaurin formula and Taylor algorithm) to get an asymptotic expansion of H_w up to order q, in the scale of functions $\{N^{-\beta}\log^\alpha N, \alpha \in \mathbb{N}, \beta \in \mathbb{N}\}$. This means that we are looking for a polynomial $p \in \mathbb{R}[X,Y]$ verifying

$$\mathrm{H}_w(N) = p(\log N, N^{-1}) + \mathrm{O}(N^{-q}). \tag{21}$$

Lemma 2 and Lemma 3 give us the following algorithm.

We use the notation $\mathbf{AE}_q(\mathrm{H}_w(N))$ for the asymptotic expansion of $\mathrm{H}_w(N)$ up to order q.

In a first time, we store the table of the asymptotic expansions, for $w \in \mathrm{Lyndon}(Y)$. For this, we proceed by recurrence on the length of w.

- If $w = y_s$, then $\mathtt{AE}_q(\mathrm{H}_w(N)) = \mathtt{AS}_q(\mathrm{H}_w(N))$, an expansion which is already known by Lemma 5, and so can be stored.
- We assume all expansions for Lyndon words of length lower or equal to L are stored. We then consider a Lyndon word of length $L+1$, $w = y_s u$. We know from Lemma 2 that the expansion of H_w is linked to the one of H_u by

$$\mathtt{AE}_q(\mathrm{H}_w(N)) = \zeta(w) - \mathtt{AS}_q\left(\sum_{i=N+1}^{\infty} \frac{\mathtt{AE}_{q-s+1}(\mathrm{H}_u(i-1))}{i^s}\right).$$

  So, there are two possibilities

  - ▷ If $u \in \mathrm{Lyndon}(Y)$ then $\mathtt{AE}_{q-s+1}\big(\mathrm{H}_u(i-1)\big)$ is assumed to be stored.
  - ▷ If $u \notin \mathrm{Lyndon}(Y)$, with the Radford decomposition, we write u as finite sum of terms $t = c\,l_1 ⧢ \cdots ⧢ l_r$, where $c \in \mathbb{Q}, l_i \in$ Lyndon, with

$$\mathtt{AE}_{q-s+1}\left(\mathrm{H}_t(i-1)\right) = c\prod_{p=1}^{r} \mathtt{AE}_{q-s+1}\left(\mathrm{H}_{l_p}(i-1)\right).$$

In a second time, if $w \notin \mathrm{Lyndon}(Y)$, as before, we use Radford decomposition and the table of the asymptotic expansion for the Lyndon words.

EXAMPLE – Let $l = y_4y_2 \in \mathrm{Lyndon}(Y)$. By Lemma 2,

$$\mathrm{H}_{4,2}(N) = \zeta(4,2) - \sum_{i=N+1}^{\infty} \frac{\mathrm{H}_2(i-1)}{i^4},$$

But $\mathrm{H}_2(i-1) = \zeta(2) - \frac{1}{i} - \frac{1}{2}\frac{1}{i^2} + \mathrm{O}\left(\frac{1}{i^3}\right)$, so

$$\begin{aligned} \mathrm{H}_{4,2}(N) &= \zeta(4,2) - \zeta(2)\sum_{i=N+1}^{\infty}\frac{1}{i^4} + \sum_{i=N+1}^{\infty}\frac{1}{i^5} \\ &+ \frac{1}{2}\sum_{i=N+1}^{\infty}\frac{1}{i^6} + \sum_{i=N+1}^{\infty}\mathrm{O}\left(\frac{1}{i^7}\right) \end{aligned}$$

Expanding the sums in N, we finally find

$$\begin{aligned} \mathrm{H}_{4,2}(N) &= \zeta(4,2) - \frac{1}{3}\frac{\zeta(2)}{N^3} + \frac{\frac{1}{2}\zeta(2) + \frac{1}{4}}{N^4} \\ &- \frac{\frac{1}{3}\zeta(2) + \frac{2}{5}}{N^5} + \mathrm{O}\left(\frac{1}{N^6}\right). \end{aligned}$$

□

EXAMPLE – Let $l = y_1y_4y_2 \notin \mathrm{Lyndon}(Y)$. The Radford decomposition of l is given by $l = y_1 ⧢ y_4y_2 - y_5y_2 - y_4y_1y_2 - y_4y_2y_1 - y_4y_3$. Using our algorithm, we find :

$$\begin{aligned} \mathrm{H}_{1,4,2}(N) &= \log(N)\zeta(4,\,2) - \zeta(4,\,1,\,2) + \gamma\zeta(4,\,2) \\ &- \zeta(5,\,2) - \zeta(4,\,2,\,1) - \zeta(4,\,3) \\ &+ \frac{1}{2}\frac{\zeta(4,\,2)}{N} - \frac{1}{12}\frac{\zeta(4,\,2)}{N^2} + \frac{1}{9}\frac{\zeta(2)}{N^3} \\ &+ \frac{-\frac{1}{24}\zeta(2) - \frac{1}{16} + \frac{1}{120}\zeta(4,\,2)}{N^4} + \mathrm{O}\left(\frac{1}{N^5}\right) \end{aligned}$$

Thanks to the table giving the relations between MZV up to weight 16[1] [7],we have the following identities

$$\begin{aligned} \zeta(4,2) &= \zeta(3)^2 - \frac{32}{105}\zeta(2)^3 \\ \zeta(4,1,2) &= \frac{5}{8}\zeta(7) + \frac{5}{2}\zeta(2)\zeta(5) - \frac{3}{2}\zeta(2)^2\zeta(3) \\ \zeta(5,2) &= -11\zeta(7) + 5\zeta(2)\zeta(5) + \frac{4}{5}\zeta(2)^2\zeta(3) \\ \zeta(4,2,1) &= -\frac{221}{16}\zeta(7) + \frac{11}{2}\zeta(2)\zeta(5) + \frac{7}{5}\zeta(2)^2\zeta(3) \\ \zeta(4,3) &= 17\zeta(7) - 10\zeta(2)\zeta(5), \end{aligned}$$

So, we deduce the reduced form of the previous expansion

$$\begin{aligned} \mathrm{H}_{1,4,2}(N) &= \log(N)(\zeta(3)^2 - \frac{32}{105}\zeta(2)^3) - \frac{32}{105}\gamma\zeta(2)^3 \\ &+ \gamma\zeta(3)^2 - 3\zeta(2)\zeta(5) - \frac{7}{10}\zeta(2)^2\zeta(3) \\ &+ \frac{115}{16}\zeta(7) + \frac{1}{2}\frac{\zeta(3)^2 - \frac{32}{105}\zeta(2)^3}{N} \\ &- \frac{1}{12}\frac{\zeta(3)^2 - \frac{32}{105}\zeta(2)^3}{N^2} + \frac{1}{9}\frac{\zeta(2)}{N^3} \\ &+ \frac{-\frac{1}{24}\zeta(2) - \frac{1}{16} + \frac{1}{120}\zeta(3)^2 - \frac{4}{1575}\zeta(2)^3}{N^4} \\ &+ \mathrm{O}\left(\frac{1}{N^5}\right) \end{aligned}$$

□

[1]This table is in agreement with the *Zagier's dimension conjecture* [15] and is available at
http://www.lifl.fr/~petitot/publis/MZV.

4.4 More examples

$$\mathrm{H}_{2,1}(N) = \zeta(3) + \frac{-\ln(N) - 1 - \gamma}{N} + \frac{\frac{1}{2}\ln(N) + \frac{1}{2}\gamma + \frac{1}{4}}{N^2} + \left(-\frac{1}{6}\gamma - \frac{5}{36} - \frac{1}{6}\ln(N)\right)\frac{1}{N^3} + \mathrm{O}\left(\frac{1}{N^4}\right)$$

$$\mathrm{H}_{3,1}(N) = \zeta(3,1) + \frac{-\frac{1}{2}\ln(N) - \frac{1}{4} - \frac{1}{2}\gamma}{N^2} + \frac{\frac{1}{2}\ln(N) + \frac{1}{2}\gamma + \frac{1}{6}}{N^3} + \left(-\frac{1}{4}\gamma - \frac{7}{48} - \frac{1}{4}\ln(N)\right)\frac{1}{N^4} + \mathrm{O}\left(\frac{\ln(N)}{N^5}\right)$$

$$\begin{aligned}\mathrm{H}_{2,1,1}(N) &= \zeta(2,1,1) + \frac{-1 - \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2)}{N} \\ &+ \frac{-\ln(N)\gamma - \gamma - \frac{1}{2}\ln^2(N) - \ln(N)}{N} \\ &+ \frac{\frac{1}{4}\ln(N) - \frac{1}{8} + \frac{1}{4}\ln^2(N) + \frac{1}{2}\ln(N)\gamma}{N^2} \\ &+ \frac{-\frac{1}{4}\zeta(2) + \frac{1}{4}\gamma + \frac{1}{4}\gamma^2}{N^2} + \Big(-\frac{5}{36}\ln(N) \\ &+ \frac{29}{216} - \frac{5}{36}\gamma - \frac{1}{6}\ln(N)\gamma - \frac{1}{12}\ln^2(N) \\ &+ \frac{1}{12}\zeta(2) - \frac{1}{12}\gamma^2\Big)\frac{1}{N^3} + \Big(\frac{1}{12}\gamma - \frac{1}{96} \\ &+ \frac{1}{12}\ln(N)\Big)\frac{1}{N^4} + \mathrm{O}\left(\frac{\ln^2(N)}{N^5}\right)\end{aligned}$$

$$\mathrm{H}_{4,1}(N) = \zeta(4,1) + \frac{-\frac{1}{3}\gamma - \frac{1}{3}\ln(N) - \frac{1}{9}}{N^3} + \frac{\frac{1}{2}\gamma + \frac{1}{2}\ln(N) + \frac{1}{8}}{N^4} + \left(-\frac{3}{20} - \frac{1}{3}\gamma - \frac{1}{3}\ln(N)\right)\frac{1}{N^5} + \mathrm{O}\left(\frac{1}{N^6}\right)$$

$$\mathrm{H}_{3,2}(N) = \zeta(3,2) - \frac{1}{2}\frac{\zeta(2)}{N^2} + \frac{\frac{1}{2}\zeta(2) + \frac{1}{3}}{N^3} + \frac{-\frac{1}{4}\zeta(2) - \frac{3}{8}}{N^4} + \mathrm{O}\left(\frac{1}{N^5}\right)$$

$$\begin{aligned}\mathrm{H}_{3,1,1}(N) &= \zeta(3,1,1) + \Big(-\frac{1}{4}(\ln(N))^2 - \frac{1}{4}\ln(N) \\ &- \frac{1}{8} - \frac{1}{2}\ln(N)\gamma - \frac{1}{4}\gamma + \frac{1}{4}\zeta(2) \\ &- \frac{1}{4}\gamma^2\Big)\frac{1}{N^2} + \Big(-\frac{1}{4}\zeta(2) + \frac{1}{6}\gamma \\ &+ \frac{1}{4}\gamma^2 - \frac{1}{9} + \frac{1}{4}(\ln(N))^2 + \frac{1}{6}\ln(N) \\ &+ \frac{1}{2}\ln(N)\gamma\Big)\frac{1}{N^3} + \mathrm{O}\left(\frac{\ln^2(N)}{N^4}\right)\end{aligned}$$

$$\begin{aligned}\mathrm{H}_{2,2,1}(N) &= \zeta(2,2,1) - \frac{\zeta(2,1)}{N} \\ &+ \frac{\frac{1}{2}\gamma + \frac{1}{2}\zeta(2,1) + \frac{1}{2}\ln(N) + \frac{3}{4}}{N^2} \\ &+ \left(-\frac{19}{36} - \frac{1}{6}\zeta(2,1) - \frac{1}{3}\gamma - 1/3\ln(N)\right)\frac{1}{N^3} \\ &+ \left(\frac{1}{24}\gamma + \frac{61}{288} + \frac{1}{24}\ln(N)\right)\frac{1}{N^4} \\ &+ \mathrm{O}\left(\frac{\ln(N)}{N^5}\right)\end{aligned}$$

$$\begin{aligned}\mathrm{H}_{2,1,1,1}(N) &= \zeta(2,1,1,1) + \Big(-\frac{1}{6}\gamma^3 - \frac{1}{3}\zeta(3) \\ &+ \frac{1}{2}\zeta(2)\gamma + \frac{1}{2}\zeta(2)\ln(N) + \frac{1}{2}\zeta(2) \\ &- \frac{1}{6}\ln(N)^3 - \frac{1}{2}(\ln(N))^2 - 1 - \ln(N) \\ &- \frac{1}{2}(\ln(N))^2\gamma - \ln(N)\gamma - \gamma - \frac{1}{2}\ln(N)\gamma^2 \\ &- \frac{1}{2}\gamma^2\Big)\frac{1}{N} + \Big(-\frac{1}{4}\zeta(2)\gamma + \frac{1}{12}\gamma^3 + \frac{1}{8}\gamma^2 \\ &- \frac{1}{8}\zeta(2) + \frac{1}{12}\ln(N)^3 - \frac{1}{8}\ln(N) - \frac{1}{16} \\ &+ \frac{1}{4}\ln(N)^2\gamma + \frac{1}{4}\ln(N)\gamma - \frac{1}{8}\gamma \\ &- \frac{1}{4}\zeta(2)\ln(N) + \frac{1}{4}\ln(N)\gamma^2 + \frac{1}{8}\ln(N)^2 \\ &+ \frac{1}{6}\zeta(3)\Big)\frac{1}{N^2} + \mathrm{O}\left(\frac{\ln^3(N)}{N^3}\right)\end{aligned}$$

$$\mathrm{H}_{5,1}(N) = \zeta(5,1) + \frac{-\frac{1}{4}\gamma - \frac{1}{4}\ln(N) - \frac{1}{16}}{N^4} + \frac{\frac{1}{2}\ln(N) + \frac{1}{2}\gamma + 1/10}{N^5} + \mathrm{O}\left(\frac{\ln(N)}{N^6}\right)$$

$$\mathrm{H}_{4,2}(N) = \zeta(4,2) - \frac{1}{3}\frac{\zeta(2)}{N^3} + \frac{\frac{1}{4} + \frac{1}{2}\zeta(2)}{N^4} + \frac{-2/5 - 1/3\zeta(2)}{N^5} + \mathrm{O}\left(\frac{1}{N^6}\right)$$

$$\begin{aligned}\mathrm{H}_{4,1,1}(N) &= \zeta(4,1,1) + \Big(\frac{1}{6}\zeta(2) - \frac{1}{6}\ln(N)^2 - \frac{1}{9}\ln(N) \\ &- \frac{1}{27} - \frac{1}{3}\ln(N)\gamma - \frac{1}{9}\gamma - \frac{1}{6}\gamma^2\Big)\frac{1}{N^3} \\ &+ \mathrm{O}\left(\frac{\ln^2(N)}{N^4}\right)\end{aligned}$$

$$\begin{aligned}\mathrm{H}_{3,2,1}(N) &= \zeta(3,2,1) - \frac{1}{2}\frac{\zeta(2,1)}{N^2} + \Big(\frac{1}{3}\gamma + \frac{4}{9} \\ &+ \frac{1}{2}\zeta(2,1) + \frac{1}{3}\ln(N)\Big)\frac{1}{N^3} + \Big(-\frac{17}{32} - \frac{3}{8}\gamma \\ &- \frac{1}{4}\zeta(2,1) - \frac{3}{8}\ln(N)\Big)\frac{1}{N^4} + \mathrm{O}\left(\frac{\ln(N)}{N^5}\right)\end{aligned}$$

$$
\begin{aligned}
\mathrm{H}_{3,1,2}(N) &= \zeta(3,1,2) + \Bigg(-\frac{1}{2}\zeta(2)\gamma + \frac{1}{2}\zeta(3) \\
&- \frac{1}{2}\zeta(2)\ln(N) - \frac{1}{4}\zeta(2) + \frac{1}{2}\zeta(2,1) \Bigg) \frac{1}{N^2} \\
&+ \Bigg(\frac{1}{6}\zeta(2) + \frac{1}{2}\zeta(2)\gamma - \frac{1}{2}\zeta(2,1) - \frac{1}{3} \\
&- \frac{1}{2}\zeta(3) + \frac{1}{2}\zeta(2)\ln(N) \Bigg) \frac{1}{N^3} + \mathrm{O}\left(\frac{\ln(N)}{N^4}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{H}_{3,1,1,1}(N) &= \zeta(3,1,1,1) + \Bigg(-\frac{1}{4}\ln(N)^2\gamma \\
&- \frac{1}{4}\ln(N)\gamma - \frac{1}{8}\gamma - \frac{1}{4}\ln(N)\gamma^2 - \frac{1}{8}\gamma^2 \\
&- \frac{1}{12}(\ln(N))^3 - \frac{1}{8}\ln(N)^2 - \frac{1}{8}\ln(N) \\
&- \frac{1}{16} + \frac{1}{4}\zeta(2)\ln(N) + \frac{1}{8}\zeta(2) + \frac{1}{4}\zeta(2)\gamma \\
&- \frac{1}{12}\gamma^3 - \frac{1}{6}\zeta(3) \Bigg) \frac{1}{N^2} + \mathrm{O}\left(\frac{\ln^3(N)}{N^3}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{H}_{2,2,1,1}(N) &= \zeta(2,2,1,1) - \frac{\zeta(2,1,1)}{N} + \Bigg(\frac{1}{2}\zeta(2,1,1) \\
&+ \frac{1}{4}\gamma^2 + \frac{3}{4}\gamma + \frac{3}{4}\ln(N) + \frac{7}{8} + \frac{1}{4}(\ln(N))^2 \\
&+ \frac{1}{2}\ln(N)\gamma - \frac{1}{4}\zeta(2) \Bigg) \frac{1}{N^2} + \mathrm{O}\left(\frac{\ln^2(N)}{N^3}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{H}_{2,1,1,1,1}(N) &= \zeta(2,1,1,1,1) + \Bigg(-1 - \gamma - \frac{1}{2}\gamma^2 \\
&- \frac{1}{24}\ln^4(N) - \frac{1}{2}\ln^2(N) - \frac{1}{24}\gamma^4 \\
&+ \frac{1}{8}\zeta(4) - \frac{1}{3}\zeta(3) - \frac{1}{2}\ln^2(N)\gamma \\
&- \frac{1}{2}\ln(N)\gamma^2 - \frac{1}{3}\zeta(3)\ln(N) - \frac{1}{3}\zeta(3)\gamma \\
&+ \frac{1}{2}\zeta(2) + \frac{1}{2}\zeta(2)\gamma - \frac{1}{6}\gamma^3 - \frac{1}{6}\ln^3(N) \\
&+ \frac{1}{4}\zeta(2)\ln^2(N) - \frac{1}{6}\ln(N)\gamma^3 \\
&- \frac{1}{4}\ln(N)^2\gamma^2 - \frac{1}{6}\ln(N)^3\gamma \\
&+ \frac{1}{2}\zeta(2)\ln(N)\gamma + \frac{1}{4}\zeta(2)\gamma^2 \\
&+ \frac{1}{2}\zeta(2)\ln(N) - \ln(N) - \frac{1}{4}\zeta(2,2) \\
&- \ln(N)\gamma \Bigg) \frac{1}{N} + \mathrm{O}\left(\frac{\ln^4(N)}{N^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{H}_{6,1}(N) &= \zeta(6,1) + \frac{-\frac{1}{5}\ln(N) - \frac{1}{25} - 1/5\gamma}{N^5} \\
&+ \frac{\frac{1}{2}\ln(N) + \frac{1}{12} + \frac{1}{2}\gamma}{N^6} \\
&+ \left(-\frac{1}{2}\gamma - \frac{13}{84} - \frac{1}{2}\ln(N) \right) \frac{1}{N^7} + \mathrm{O}\left(\frac{\ln(N)}{N^8}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{H}_{5,2}(N) &= \zeta(5,2) - \frac{1}{4}\frac{\zeta(2)}{N^4} + \frac{\frac{1}{2}\zeta(2) + \frac{1}{5}}{N^5} \\
&+ \frac{-\frac{5}{12} - \frac{5}{12}\zeta(2)}{N^6} + \frac{\frac{23}{84}}{N^7} + \mathrm{O}\left(\frac{1}{N^8}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{H}_{5,1,1}(N) &= \zeta(5,1,1) + \Bigg(-\frac{1}{4}\ln(N)\gamma - \frac{1}{16}\gamma \\
&- \frac{1}{8}\ln^2(N) - \frac{1}{16}\ln(N) - \frac{1}{64} - \frac{1}{8}\gamma^2 \\
&+ \frac{1}{8}\zeta(2) \Bigg) \frac{1}{N^4} + \mathrm{O}\left(\frac{\ln^2(N)}{N^5}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{H}_{4,3}(N) &= \zeta(4,3) - \frac{1}{3}\frac{\zeta(3)}{N^3} + \frac{1}{2}\frac{\zeta(3)}{N^4} + \frac{-\frac{1}{3}\zeta(3) + 1/10}{N^5} \\
&- \frac{\frac{1}{6}}{N^6} + \frac{\frac{1}{28} + \frac{1}{6}\zeta(3)}{N^7} + \mathrm{O}\left(\frac{1}{N^8}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{H}_{4,2,1}(N) &= \zeta(4,2,1) - \frac{1}{3}\frac{\zeta(2,1)}{N^3} + \Bigg(\frac{1}{4}\gamma + \frac{1}{2}\zeta(2,1) \\
&+ \frac{5}{16} + \frac{1}{4}\ln(N) \Bigg) \frac{1}{N^4} + \Bigg(-2/5\gamma - \frac{53}{100} \\
&- 2/5\ln(N) - \frac{1}{3}\zeta(2,1) \Bigg) \frac{1}{N^5} + \mathrm{O}\left(\frac{\ln(N)}{N^6}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{H}_{4,1,2}(N) &= \zeta(4,1,2) + \Bigg(-\frac{1}{3}\zeta(2)\ln(N) - \frac{1}{9}\zeta(2) \\
&+ \frac{1}{3}\zeta(3) - \frac{1}{3}\zeta(2)\gamma + \frac{1}{3}\zeta(2,1) \Bigg) \frac{1}{N^3} \\
&+ \mathrm{O}\left(\frac{\ln(N)}{N^4}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{H}_{4,1,1,1}(N) &= \zeta(4,1,1,1) + \Bigg(-\frac{1}{9}\zeta(3) - \frac{1}{18}\gamma^3 \\
&+ \frac{1}{6}\zeta(2)\gamma - \frac{1}{6}\ln^2(N)\gamma - \frac{1}{9}\ln(N)\gamma - \frac{1}{27}\gamma \\
&- \frac{1}{18}\ln^3 N - \frac{1}{18}(\ln(N))^2 - \frac{1}{27}\ln(N) \\
&- \frac{1}{81} + \frac{1}{6}\zeta(2)\ln(N) + \frac{1}{18}\zeta(2) - \frac{1}{6}\ln(N)\gamma^2 \\
&- \frac{1}{18}\gamma^2 \Bigg) \frac{1}{N^3} + \mathrm{O}\left(\frac{\ln^3(N)}{N^3}\right)
\end{aligned}
$$

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