Algorithmic and Combinatoric Aspects of Multiple Harmonic Sums

C. Costermans J.Y. Enjalbert Hoang Ngoc Minh

University Lille 2

Analysis of Algorithms
Barcelona, June 06-10, 2005

Outline

1. Motivation
2. Generating series
   - Ordinary G.S. of $\{\mathrm{H}_w(N), N \geq 0\}$
   - Noncommutative G.S. of $\{\mathrm{P}_w, w \in X^*\}$
3. Computing Asymptotic Expansion
   - Singular expansion
   - Asymptotic expansion

Outline

1. Motivation

2. Generating series
   - Ordinary G.S. of $\{\mathrm{H}_w(N), N \geq 0\}$
   - Noncommutative G.S. of $\{\mathrm{P}_w, w \in X^*\}$

3. Computing Asymptotic Expansion
   - Singular expansion
   - Asymptotic expansion

Multiple Harmonic Sums

- Generalized Harmonic Numbers

$$\mathrm{H}_r(N) = \sum_{n=1}^{N} \frac{1}{n^r} \qquad N \in \mathbb{N}, r \geq 0$$

Extended to compositions $\underline{\mathbf{s}} = (s_1, \ldots, s_r)$

$$\mathrm{H}_{\underline{\mathbf{s}}}(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}$$

- Appear in the study of probabilities (quadtrees [Flajolet et al., 93], maxima in hypercubes [Devroye et al., 05]), and also in quantum physics, knot theory...etc

Multiple Harmonic Sums

- Generalized Harmonic Numbers

$$\mathrm{H}_r(N) = \sum_{n=1}^{N} \frac{1}{n^r} \qquad N \in \mathbb{N}, r \geq 0$$

Extended to compositions $\underline{\mathbf{s}} = (s_1, \ldots, s_r)$

$$\mathrm{H}_{\underline{\mathbf{s}}}(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}$$

- Appear in the study of probabilities (quadtrees [Flajolet et al., 93], maxima in hypercubes [Devroye et al., 05]), and also in quantum physics, knot theory...etc

Our goal

- Asymptotic behaviour of $\mathrm{H}_{\underline{s}}(N)$?
- Linked the behaviour of $\mathrm{P}_{\underline{s}}(z) = \sum_{N \geq 0} \mathrm{H}_{\underline{s}}(N) z^N$, as $z \to 1$. But

$$\mathrm{P}_{\underline{s}}(z) = \frac{1}{1-z} \mathrm{Li}_{\underline{s}}(z)$$

Polylogarithm ?

- So ? Let us use some results on the noncommutative generating series

$$\mathrm{P}(z) = \sum_{w \in X^*} \mathrm{P}_w(z) w.$$

Our goal

- Asymptotic behaviour of $\mathrm{H}_{\underline{\mathbf{s}}}(N)$?
- Linked the behaviour of $\mathrm{P}_{\underline{\mathbf{s}}}(z) = \sum_{N\geq 0} \mathrm{H}_{\underline{\mathbf{s}}}(N) z^N$, as $z \rightarrow 1$. But

$$\mathrm{P}_{\underline{\mathbf{s}}}(z) = \frac{1}{1-z}\,\mathrm{Li}_{\underline{\mathbf{s}}}(z)$$

Polylogarithm ?

- So ? Let us use some results on the noncommutative generating series

$$\mathrm{P}(z) = \sum_{w \in X^*} \mathrm{P}_w(z) w.$$

Outline

1. Motivation
2. Generating series
   - Ordinary G.S. of $\{\mathrm{H}_w(N), N \geq 0\}$
   - Noncommutative G.S. of $\{\mathrm{P}_w, w \in X^*\}$
3. Computing Asymptotic Expansion
   - Singular expansion
   - Asymptotic expansion

Definition

Definition

For $w \in X^*$, we put $\mathrm{P}_w(z) = \dfrac{\mathrm{Li}_w(z)}{1-z}$.

Proposition

Let $w \in X^*x_1$. Then $\mathrm{P}_w(z)$ is the ordinary generating series of $\{\mathrm{H}_w(N), N \geq 0\}$:

$$\mathrm{P}_w(z) = \sum_{N \geq 0} \mathrm{H}_w(N) z^N$$

Noncommutative generating series

Let $\mathrm{L}(z) = \sum_{w \in X^*} \mathrm{Li}_w(z) w$ and $\mathrm{P}(z) = \sum_{w \in X^*} \mathrm{P}_w(z) w.$

Theorem (Factorization theorem)

$$\mathrm{P}(z) = \frac{z}{1-z}[\sigma \mathrm{P}(1-z)] \prod_{l \in Lyn(X) \setminus \{x_0, x_1\}}^{\searrow} e^{\zeta(S_l) Q_l},$$

where σ is the morphism defined by $\sigma(x_0) = -x_1, \sigma(x_1) = -x_0$.

This is due to the fact that [Hoang et al.,99]

$$\mathrm{L}(z) = [\sigma \mathrm{L}(1-z)] \prod_{l \in Lyn(X) \setminus \{x_0, x_1\}}^{\searrow} e^{\zeta(S_l) Q_l}.$$

Lyndon word? Q_l, S_l?

Noncommutative generating series

Let $\mathrm{L}(z) = \sum_{w \in X^*} \mathrm{Li}_w(z) w$ and $\mathrm{P}(z) = \sum_{w \in X^*} \mathrm{P}_w(z) w.$

Theorem (Factorization theorem)

$$\mathrm{P}(z) = \frac{z}{1-z}[\sigma\mathrm{P}(1-z)] \prod_{l \in Lyn(X) \setminus \{x_0, x_1\}}^{\searrow} e^{\zeta(S_l) Q_l},$$

where σ is the morphism defined by $\sigma(x_0) = -x_1, \sigma(x_1) = -x_0$.

This is due to the fact that [Hoang et al.,99]

$$\mathrm{L}(z) = [\sigma\mathrm{L}(1-z)] \prod_{l \in Lyn(X) \setminus \{x_0, x_1\}}^{\searrow} e^{\zeta(S_l) Q_l}.$$

Lyndon word? Q_l, S_l?

Example

For example,

$$\begin{aligned}\mathrm{P}_{2,1}(z) &= -\frac{z}{1-z}\mathrm{P}_3(1-z) + \frac{z}{1-z}\log(1-z)\mathrm{P}_2(1-z) \\ &- \frac{1}{2}\frac{z}{1-z}\log^2(1-z)\mathrm{P}_1(1-z) + \frac{\zeta(3)}{1-z}.\end{aligned}$$

Then, we deduce the *singular expansion* of $\mathrm{P}_{2,1}$ around $z = 1$.

Outline

1. Motivation

2. Generating series
   - Ordinary G.S. of $\{\mathrm{H}_w(N), N \geq 0\}$
   - Noncommutative G.S. of $\{\mathrm{P}_w, w \in X^*\}$

3. Computing Asymptotic Expansion
   - Singular expansion
   - Asymptotic expansion

Main Theorem

Theorem

Let $\mathcal{C} = \mathbb{C}\left[z, \frac{1}{z}, \frac{1}{1-z}\right]$ *and* $g \in \mathcal{C}[(\mathrm{P}_w)_{w \in X^*x_1}]$. *There exist* $a_j \in \mathbb{C}, \alpha_j \in \mathbb{Z}$ *and* $\beta_j \in \mathbb{N}$ *such that*

$$g(z) \sim \sum_{j=0}^{+\infty} a_j (1-z)^{\alpha_j} \log^{\beta_j}(1-z), \quad \textit{for} \quad z \to 1.$$

Therefore, there exist $b_i \in \mathbb{C}, \eta_i \in \mathbb{Z}$ *and* $\kappa_i \in \mathbb{N}$ *such that*

$$[z^n]g(z) \sim \sum_{i=0}^{+\infty} b_i n^{\eta_i} \log^{\kappa_i}(n), \quad \textit{for} \quad n \to \infty.$$

Proof(1)

- First : reduction to the case $f(z) = \mathrm{P}_w(z)$.
  Factorization of $\mathrm{P} \Rightarrow \mathrm{P}_w(z) = \sum_{finite} \lambda_u \mathrm{P}_u(1-z), u \in X^*x_1$.
- For $u \in X^*x_1 \cup \{x_0\}$, we have

$$\mathrm{P}_u(1-z) = \sum_{n\geq 0} \mathrm{H}_u(n)(1-z)^n$$

$$\mathrm{P}_{x_0}(z) = \frac{\log(1-z)}{z}.$$

$\Rightarrow$ The first expansion follows.

Proof(1)

- First : reduction to the case $f(z) = \mathrm{P}_w(z)$.
  Factorization of $\mathrm{P} \Rightarrow \mathrm{P}_w(z) = \sum_{finite} \lambda_u \mathrm{P}_u(1-z), u \in X^*x_1$.
- For $u \in X^*x_1 \cup \{x_0\}$, we have

$$\mathrm{P}_u(1-z) = \sum_{n\geq 0} \mathrm{H}_u(n)(1-z)^n$$

$$\mathrm{P}_{x_0}(z) = \frac{\log(1-z)}{z}.$$

$\Rightarrow$ The first expansion follows.

Example

From

$$\begin{aligned} \mathrm{P}_{2,1}(z) &= -\frac{z}{1-z}\mathrm{P}_3(1-z) + \frac{z}{1-z}\log(1-z)\mathrm{P}_2(1-z) \\ &- \frac{1}{2}\frac{z}{1-z}\log^2(1-z)\mathrm{P}_1(1-z) + \frac{\zeta(3)}{1-z}, \end{aligned}$$

we deduce

$$\begin{aligned} \mathrm{P}_{2,1}(z) &= \frac{\zeta(3)}{1-z} + \log(1-z) - 1 - \frac{\log^2(1-z)}{2} \\ &+ \frac{1-z}{4}\left(-\log^2(1-z) + \log(1-z)\right) + \mathrm{O}(|1-z|). \end{aligned}$$

Proof(2)

- Note that $(1-z)^\alpha \log^\beta(1-z) = (-1)^\beta \beta!(1-z)^{\alpha+1}\mathrm{P}_{y_1^\beta}(z)$.
  But $[z^n]\mathrm{P}_{y_1^\beta}(z) = \mathrm{H}_{y_1^\beta}(n) = \mathrm{H}_{1,1,\cdots,1}(n)$.

Proposition

$\mathrm{H}_{y_1^k}$ *is an algebraic combination of* $\{\mathrm{H}_r\}_{1\le r\le k}$, *which are algebraically independent.*

- Expansion of $\mathrm{H}_r(n)$ computable by Euler-MacLaurin
- Then, action of $1-z$ over P_w

$$
\begin{aligned}
[z^n](1-z)\mathrm{P}_w(z) &= \mathrm{H}_w(n) - \mathrm{H}_w(n-1), \\
[z^n]\frac{\mathrm{P}_w(z)}{1-z} &= \sum_{k=0}^{n} \mathrm{H}_w(k).
\end{aligned}
$$

Example

$$
\begin{aligned}
\mathrm{P}_{2,1}(z) &= \frac{\zeta(3)}{1-z} + \log(1-z) - 1 - \frac{\log^2(1-z)}{2} \\
&+ (1-z)\left(-\frac{\log^2(1-z)}{4} + \frac{\log(1-z)}{4}\right) + \mathrm{O}(|1-z|),
\end{aligned}
$$

But

$$
\begin{aligned}
[z^N]\frac{\zeta(3)}{1-z} &= \zeta(3) \\
[z^N]\log(1-z) &= -\frac{1}{N},
\end{aligned}
$$

Example(2)

$$[z^N]\frac{\log^2(1-z)}{2} = [z^N]\frac{2!(1-z)\mathrm{P}_{y_1^2}(z)}{2} = [z^N](1-z)\mathrm{P}_{y_1^2}(z) = \mathrm{H}_{y_1^2}(N) - \mathrm{H}_{y_1^2}(N-1)$$

But

$$\mathrm{H}_{y_1^2}(N) = \frac{1}{2}[\mathrm{H}_1^2(N) - \mathrm{H}_2(N)].$$

So, we finally find :

$$[z^N]\mathrm{P}_{2,1}(z) = \mathrm{H}_{2,1}(N) = \zeta(3) - \frac{\log(N)+1+\gamma}{N} + \frac{1}{2}\frac{\log(N)}{N^2} + \mathrm{O}\left(\frac{1}{N^2}\right).$$

Thank you for your attention

Polylogarithms and Euler-ZAgier sums : definitions

- For k a positive integer,

$$\mathrm{Li}_k(z) = \sum_{n>0} \frac{z^n}{n^k}.$$

- For $\underline{\mathbf{s}} = (s_1, \ldots, s_r)$, and for $|z| < 1$ we define

$$\mathrm{Li}_{\underline{\mathbf{s}}}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}.$$

For $s_1 > 1$, by an Abel's theorem, we have :

$$\lim_{z \to 1} \mathrm{Li}_{\underline{\mathbf{s}}}(z) = \lim_{N \to +\infty} \mathrm{H}_{\underline{\mathbf{s}}}(N) = \zeta(\underline{\mathbf{s}}).$$

Encoding by words

Encoding for $\underline{\mathbf{s}}$:

$$\underline{\mathbf{s}} = (s_1, \ldots, s_r) \longleftrightarrow w = x_0^{s_1-1} x_1 \cdots x_0^{s_r-1} x_1 \in X^*,$$

where $X = \{x_0, x_1\}$.

$$\mathrm{Li}_{\underline{\mathbf{s}}}(z) = \mathrm{Li}_w(z),$$

and in the same way, we denote

$$\mathrm{H}_{\underline{\mathbf{s}}}(N) = \mathrm{H}_w(N), \quad \text{and} \quad \zeta(\underline{\mathbf{s}}) = \zeta(w).$$

We extend also the definition of Li_w by putting $\mathrm{Li}_{x_0}(z) = \log(z)$.

Back

Lyndon words

- X^* totally ordered by putting : $x_0 < x_1$.
- l Lyndon word iff $l = uv, v \neq \epsilon \Rightarrow l < v$

Example

Set of Lyndon words over X, of length ≤ 4 :

$$Lyn(X) = \{x_0, x_1, x_0^2 x_1, x_0 x_1^2, x_0^3 x_1, x_0^2 x_1^2, x_0 x_1^3, \ldots\}$$

Back

P.B.W. Basis [Reutenauer, 93]

- For $l \in Lyn(X), l = uv, u, v \in Lyn(X)$ and v as long as possible

$$\begin{cases} \mathcal{Q}_l & = & [\mathcal{Q}_u, \mathcal{Q}_v] = \mathcal{Q}_u \mathcal{Q}_v - \mathcal{Q}_v \mathcal{Q}_u \\ \mathcal{Q}_x & = & x \quad \text{if } x \in X, \end{cases}$$

- For $l = xw \in Lyn(X), x \in X$, $w = l_1^{\alpha_1} l_2^{\alpha_2} \ldots l_k^{\alpha_k}, l_1 > l_2 > \cdots > l_k$.

$$\mathcal{S}_l = x \frac{\mathcal{S}_{l_1}^{\sqcup\!\sqcup\, \alpha_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup \mathcal{S}_{l_k}^{\sqcup\!\sqcup\, \alpha_k}}{\alpha_1! \alpha_2! \ldots \alpha_k!},$$

where $\sqcup\!\sqcup$ denotes the shuffle product on words.

P.B.W. Basis [Reutenauer, 93]

- For $l \in Lyn(X), l = uv, u, v \in Lyn(X)$ and v as long as possible

$$\begin{cases} \mathcal{Q}_l &= [\mathcal{Q}_u, \mathcal{Q}_v] = \mathcal{Q}_u \mathcal{Q}_v - \mathcal{Q}_v \mathcal{Q}_u \\ \mathcal{Q}_x &= x \quad \text{if } x \in X, \end{cases}$$

- For $l = xw \in Lyn(X), x \in X$,
$w = l_1^{\alpha_1} l_2^{\alpha_2} \ldots l_k^{\alpha_k}, l_1 > l_2 > \cdots > l_k$.

$$\mathcal{S}_l = x \frac{\mathcal{S}_{l_1}^{\sqcup\!\sqcup\, \alpha_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup\, \mathcal{S}_{l_k}^{\sqcup\!\sqcup\, \alpha_k}}{\alpha_1! \alpha_2! \ldots \alpha_k!},$$

where $\sqcup\!\sqcup$ denotes the shuffle product on words.

Lyndon words, bracket forms and dual basis

l	$\mathcal{Q}_l$	$\mathcal{S}_l$
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0, x_1]$	x_0x_1
$x_0{}^2x_1$	$[x_0, [x_0, x_1]]$	$x_0{}^2x_1$
$x_0x_1{}^2$	$[[x_0, x_1], x_1]$	$x_0x_1{}^2$
$x_0{}^3x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0{}^3x_1$
$\vdots$	$\vdots$	$\vdots$
$x_0{}^3x_1{}^3$	$[x_0, [x_0, [[[x_0, x_1], x_1], x_1]]]$	$x_0{}^3x_1{}^3$
$x_0{}^2x_1x_0x_1{}^2$	$[x_0, [[x_0, x_1], [[x_0, x_1], x_1]]]$	$3x_0{}^3x_1{}^3 + x_0{}^2x_1x_0x_1{}^2$

Back

Going further

$$\mathrm{H}_1(N) \sim \log N + \gamma - \sum_{k=1}^{+\infty} \frac{B_k}{k} \frac{1}{N^k},$$

$$\mathrm{H}_r(N) \sim \zeta(r) - \frac{1}{(r-1)N^{r-1}} - \sum_{k=r}^{+\infty} \frac{B_{k-r+1}}{k-r+1} \binom{k-1}{r-1} \frac{1}{N^k}$$

And

$$\mathrm{H}_{y_1^k} = \sum_{a_1+2a_2+\ldots+ka_k=k} (-1)^{k-\sum a_i} \frac{\mathrm{H}_1^{a_1} \cdots \mathrm{H}_k^{a_k}}{1^{a_1} a_1! \cdots k^{a_k} a_k!}$$

Outline

Shuffle products

The *shuffle* (resp. *stuffle*) of $u = au'$ and $v = bv' \in X^*$ (resp. $u = y_i u'$ and $v = y_j v' \in Y^*$) is defined by

$$\begin{aligned} \epsilon \sqcup\!\sqcup u &= u \sqcup\!\sqcup \epsilon = u \text{ and} \\ u \sqcup\!\sqcup v &= a(u' \sqcup\!\sqcup v) + b(u \sqcup\!\sqcup v'), \end{aligned}$$

$$\text{resp.} \quad \begin{aligned} \epsilon \uplus u &= u \uplus \epsilon = u \text{ and} \\ u \uplus v &= y_i(u' \uplus v) + y_j(u \uplus v') + y_{i+j}(u' \uplus v'). \end{aligned}$$

For example,

- $x_0x_1 \sqcup\!\sqcup x_1 = x_1x_0x_1 + 2x_0x_1^2$ and $y_2 \uplus y_1 = y_1y_2 + y_2y_1 + y_3$,
- $x_0x_1 \sqcup\!\sqcup x_0x_1 = 2x_0x_1x_0x_1 + 4x_0x_1^2$ and $y_2y_1 \uplus y_3 = y_2y_1y_3 + y_2y_3y_1 + y_3y_2y_1 + y_2y_4 + y_5y_1$.

Shuffle products

The *shuffle* (resp. *stuffle*) of $u = au'$ and $v = bv' \in X^*$ (resp. $u = y_i u'$ and $v = y_j v' \in Y^*$) is defined by

$$\begin{aligned} \epsilon \sqcup\!\sqcup u &= u \sqcup\!\sqcup \epsilon = u \text{ and} \\ u \sqcup\!\sqcup v &= a(u' \sqcup\!\sqcup v) + b(u \sqcup\!\sqcup v'), \end{aligned}$$

$$\text{resp.} \quad \begin{aligned} \epsilon \uplus u &= u \uplus \epsilon = u \text{ and} \\ u \uplus v &= y_i(u' \uplus v) + y_j(u \uplus v') + y_{i+j}(u' \uplus v'). \end{aligned}$$

For example,

- $x_0x_1 \sqcup\!\sqcup x_1 = x_1x_0x_1 + 2x_0x_1^2$ and $y_2 \uplus y_1 = y_1y_2 + y_2y_1 + y_3$,
- $x_0x_1 \sqcup\!\sqcup x_0x_1 = 2x_0x_1x_0x_1 + 4x_0x_1^2$ and $y_2y_1 \uplus y_3 = y_2y_1y_3 + y_2y_3y_1 + y_3y_2y_1 + y_2y_4 + y_5y_1$.

Shuffle products

The *shuffle* (resp. *stuffle*) of $u = au'$ and $v = bv' \in X^*$ (resp. $u = y_i u'$ and $v = y_j v' \in Y^*$) is defined by

$$\begin{array}{rcl} \epsilon ⧢ u & = & u ⧢ \epsilon = u \text{ and} \\ u ⧢ v & = & a(u' ⧢ v) + b(u ⧢ v'), \end{array}$$

$$\text{resp.} \quad \begin{array}{rcl} \epsilon \uplus u & = & u \uplus \epsilon = u \text{ and} \\ u \uplus v & = & y_i(u' \uplus v) + y_j(u \uplus v') + y_{i+j}(u' \uplus v'). \end{array}$$

For example,

- $x_0x_1 ⧢ x_1 = x_1x_0x_1 + 2x_0x_1^2$ and
  $y_2 \uplus y_1 = y_1y_2 + y_2y_1 + y_3$,
- $x_0x_1 ⧢ x_0x_1 = 2x_0x_1x_0x_1 + 4x_0x_1^2$ and
  $y_2y_1 \uplus y_3 = y_2y_1y_3 + y_2y_3y_1 + y_3y_2y_1 + y_2y_4 + y_5y_1$.

Justification of the stuffle product

$$\sum_{n=1}^{N}\frac{1}{n^{s_1}}\sum_{m=1}^{N}\frac{1}{m^{s_2}} = \sum_{N\geq n>m>0}\frac{1}{n^{s_1}m^{s_2}}+\sum_{N\geq m>n>0}\frac{1}{m^{s_2}n^{s_1}} + \sum_{N\geq n>0}\frac{1}{n^{s_1+s_2}}$$

$$\begin{aligned}\mathrm{H}_{s_1}(N)\mathrm{H}_{s_2}(N) &= \mathrm{H}_{s_1,s_2}(N)+\mathrm{H}_{s_2,s_1}(N)+\mathrm{H}_{s_1+s_2}(N)\\ y_{s_1} \uplus y_{s_2} &= y_{s_1,s_2}+y_{s_2,s_1}+y_{s_1+s_2}\end{aligned}$$

With the convention $\mathrm{H}_{\underline{s}} = \mathrm{H}_w$,

Theorem (Hoang, 03)

Let $\mathcal{H}_{\mathbb{C}} = (\mathrm{span}_{\mathbb{C}}(\mathrm{H}_w \mid w \in Y^*), \cdot)$. *Then* $\mathcal{H}_{\mathbb{C}} \simeq (\mathbb{C}\langle Y\rangle, \uplus)$

Justification of the stuffle product

$$\sum_{n=1}^{N}\frac{1}{n^{s_1}}\sum_{m=1}^{N}\frac{1}{m^{s_2}} = \sum_{N\geq n>m>0}\frac{1}{n^{s_1}m^{s_2}} + \sum_{N\geq m>n>0}\frac{1}{m^{s_2}n^{s_1}} + \sum_{N\geq n>0}\frac{1}{n^{s_1+s_2}}$$

$$\mathrm{H}_{s_1}(N)\mathrm{H}_{s_2}(N) = \mathrm{H}_{s_1,s_2}(N)+\mathrm{H}_{s_2,s_1}(N)+\mathrm{H}_{s_1+s_2}(N)$$

$$y_{s_1} \uplus y_{s_2} = y_{s_1,s_2}+y_{s_2,s_1}+y_{s_1+s_2}$$

With the convention $\mathrm{H}_{\underline{\mathbf{s}}} = \mathrm{H}_w$,

Theorem (Hoang, 03)

Let $\mathcal{H}_{\mathbb{C}} = (\mathrm{span}_{\mathbb{C}}(\mathrm{H}_w \mid w \in Y^*), \cdot)$. *Then* $\mathcal{H}_{\mathbb{C}} \simeq (\mathbb{C}\langle Y\rangle, \uplus)$

Polylogarithms : properties (1)

$$\underline{\mathbf{s}} = (s_1, \ldots, s_r) \longleftrightarrow u = x_0^{s_1-1}x_1 \cdots x_0^{s_r-1}x_1 \in X^*$$

Putting $\omega_0 = \frac{dz}{z}$ and $\omega_1 = \frac{dz}{1-z}$, we have

$$\mathrm{Li}_{\underline{\mathbf{s}}}(z) = \mathrm{Li}_u(z) = \int_{0 \rightsquigarrow z} \omega_0^{s_1-1}\omega_1 \cdots \omega_0^{s_r-1}\omega_1.$$

$\Rightarrow$ Allows to extend the definition of Li_u over X^* with the definition

$$\mathrm{Li}_{x_0}(z) = \log(z).$$

Polylogarithms : properties (2)

Theorem (Hoang et al.,00)

The $\mathbb{C}$*-algebra of* $\{\mathrm{Li}_w, w \in X^*\}$ *is isomorphic to* $(\mathbb{C}\langle X\rangle, ⧢)$*.*

Examples

Since $x_0x_1 ⧢ x_1 = x_1x_0x_1 + 2x_0x_1^2$, we get :

$$\mathrm{Li}_2 \mathrm{Li}_1 = \mathrm{Li}_{1,2} + 2\,\mathrm{Li}_{2,1}$$

From $x_0x_1 ⧢ x_0x_1 = 2x_0x_1x_0x_1 + 4x_0x_1^2$, we get :

$$\mathrm{Li}_2^2 = 2\,\mathrm{Li}_{2,2} + 4\,\mathrm{Li}_{2,1}$$