Some results *à l'Abel* obtained by use of techniques *à la Hopf*[1]

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Abstract

In this work, we obtain some results *à l'Abel* dealing with noncommutative generating series of polylogarithms and multiple harmonic sums, by using techniques *à la Hopf*. In particular, this enables to explicit generalized Euler constants associated to divergent polyzêtas and to extract the constant part of (commutative and noncommutative) generating series of all polyzêtas.

Keywords: asymptotic expansion, generating series, multiple harmonic sums, polylogarithms, polyzêtas

1 Introduction

Let us consider the alphabet $Y = \{y_i\}_{i\in\mathbb{N}_+}$. To each word $w = y_{s_1}\ldots y_{s_r}$ of the monoid Y^*, we associate the multiple harmonic sum $\mathrm{H}_w(N)$ and the polylogarithm $\mathrm{Li}_w(z)$

$$\mathrm{H}_w(N) = \sum_{N\geq n_1>\ldots>n_r>0} \frac{1}{n_1^{s_1}\ldots n_r^{s_r}}, \quad \mathrm{Li}_w(z) = \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}}. \qquad (1)$$

For $0 \leq N < r, \mathrm{H}_w(N) = 0$ and for the empty word ϵ, we put $\mathrm{H}_\epsilon(N) = 1$, for any $N \geq 0$. For $w \in Y^*\setminus y_1Y^*$, the limits $\lim_{z\to 1}\mathrm{Li}_w(z)$ and $\lim_{N\to\infty}\mathrm{H}_w(N)$

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exist and, by an Abel theorem, are equal to the convergent polyzêta $\zeta(w)$

$$\zeta(w) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, \qquad s_1 > 1. \tag{2}$$

In other cases, *i.e.* for $w = y_s w'$, the associated polylogarithm Li_w and polyzêta $\zeta(w)$ can be considered respectively as a polylogarithmic generating series and as Dirichlet series

$$\mathrm{Li}_{y_s w'}(z) = \sum_{N>0} \frac{p_N}{N^s} z^N \quad \text{and} \quad \zeta(y_s w') = \sum_{N>0} \frac{p_N}{N^s} \tag{3}$$

with $p_N = \mathrm{H}_{w'}(N-1)$. Both series can be obtained from the following generating series

$$\mathrm{P}_{w'}(z) = \sum_{N=0}^{\infty} \mathrm{H}_{w'}(N) z^N = \sum_{N \geq 0} p_{N+1} z^N, \tag{4}$$

respectively by the polylogarithmic transform and by the Mellin transform [1]

$$\mathrm{Li}_{y_s w'}(z) = \int_0^{\infty} \frac{\mathrm{P}_{w'}(ze^{-u})}{\Gamma(s)} \frac{du}{u^{1-s}} \quad \text{and} \quad \zeta(y_s w') = \int_0^{\infty} \frac{\mathrm{P}_{w'}(e^{-u})}{\Gamma(s)} \frac{du}{u^{1-s}}. \tag{5}$$

The generating series $\mathrm{P}_{w'}$ can also be expressed using the polylogarithm :

$$\mathrm{P}_{w'}(z) = (1-z)^{-1} \, \mathrm{Li}_{w'}(z). \tag{6}$$

The knowledge of the singular expansion of $\mathrm{P}_{w'}$ in the scale $\{(1-z)^a \log^b(1-z)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$ enables then to get, on the first hand the asymptotic behaviour, as $N \to \infty$, of its Taylor coefficients $\mathrm{H}_{w'}(N)$ in the scale $\{N^\alpha \log^\beta N\}_{\alpha \in \mathbb{Z}, \beta \in \mathbb{N}}$. Then, to deduce the behaviour of $\mathrm{H}_w(N)$, since $\mathrm{H}_w(N) = \sum_{i=1}^N \mathrm{H}_{w'}(i-1)/i^s$. This gives on the other hand, through a tauberian theorem, the singular expansion of Dirichlet series $\zeta(y_s w')$ considered then as a function of the complex variable s.

Both studies lead to apply another Abel theorem dealing with Dirichlet series [2]. Indeed, let us consider the partial sum S_N of coefficients of the ordinary generating series $\mathrm{P}_{w'}(z)$,

$$S_N = \sum_{i=1}^{N} p_{i+1} = \sum_{i=1}^{N} \mathrm{H}_{w'}(i). \tag{7}$$

If S_N admits a singular expansion of the following type

$$S_N = \sum_{j=1}^{k} B_j N^{\sigma_j} \log^{\alpha_j} N + O(N^{\beta}), \tag{8}$$

where, for all $j = 1, .., k, B_j$ is an arbitrary complex number, σ_j, α_j are arbitrary integers, and β is an integer such that $\beta > \sigma_k$, then the Dirichlet series $\zeta(y_s w')$ is convergent for $s > 1$ and even regular except in $\sigma_1, \ldots, \sigma_k$ which are its logarithmic singularities.

In order to adapt automatically these Abel techniques to polylogarithms $\{\mathrm{Li}_w\}_{w \in Y^*}$ and to multiple harmonic sums $\{\mathrm{H}_w\}_{w \in Y^*}$, we consider the *non-commutative generating series*

$$\Lambda(z) = \sum_{w \in Y^*} \mathrm{Li}_w(z)\, w \quad \text{and} \quad \mathrm{H}(N) = \sum_{w \in Y^*} \mathrm{H}_w(N)\, w. \tag{9}$$

Through algebraic combinatoric [3] and elements of topology of formal series in noncommutative variables [4], we show in Section 2.2 the existence of formal series over Y, Z_1 and Z_2 in non commutative variables with constant coefficients, such that

$$\lim_{z \to 1} \exp\left[y_1 \log \frac{1}{1-z}\right] \Lambda(z) = Z_1 \text{ and} \tag{10}$$

$$\lim_{N \to \infty} \exp\left[\sum_{k \geq 1} \mathrm{H}_{y_k}(N) \frac{(-y_1)^k}{k}\right] \mathrm{H}(N) = Z_2.$$

Moreover, we have $Z_1 = Z_2$, both standing for the noncommutative generating series of all convergent polyzêtas $\{\zeta(w)\}_{w \in Y^* \backslash y_1 Y^*}$, (as shown by the factorized form). This enables, in particular, to explicit generalized Euler constants associated to divergent polyzêtas $\{\zeta(w)\}_{w\in y_1Y^*}$ and to extract the constant part of generating series (commutative and noncommutative) of all polyzêtas.

Techniques presented in this paper can be applied to other fields, like polysystems occuring in physical problems, and enable to make the calculations easier. To illustrate this, we present in appendix some results to compute, thanks to such techniques, the solution of a linear differential system, with three singularities, that can be supposed to be $\{0, 1, \infty\}$, after an homographic transformation.

2 Polylogarithm and harmonic sum

2.1 Algebraic properties

2.1.1 Symmetric functions and harmonic sums

Let $\{t_i\}_{i\in\mathbb{N}_+}$ be an infinite set of variables. The elementary symmetric functions λ_k and the sums of powers ψ_k are defined by

$$\lambda_k(\underline{t}) = \sum_{n_1>\ldots>n_k>0} t_{n_1}\ldots t_{n_k} \quad \text{and} \quad \psi_k(\underline{t}) = \sum_{n>0} t_n^k. \tag{11}$$

They are respectively coefficients of the following generating functions

$$\lambda(\underline{t}|z) = \sum_{k>0} \lambda_k(\underline{t})z^k = \prod_{i\geq 1}(1+t_iz) \text{ and} \tag{12}$$

$$\psi(\underline{t}|z) = \sum_{k>0} \psi_k(\underline{t})z^{k-1} = \sum_{i\geq 1} \frac{t_i}{1-t_iz}.$$

These generating functions satisfy a Newton identity

$$d/dz \log \lambda(\underline{t}|z) = \psi(\underline{t}|-z). \tag{13}$$

The fundamental theorem from symmetric functions theory asserts that the $\{\lambda_k\}_{k\geq 0}$ are linearly independent, and remarkable identities give (putting $\lambda_0 = 1$) :

$$k!\lambda_k \;=\; (-1)^k \sum_{\substack{s_1,\ldots,s_k>0\\ s_1+\ldots+ks_k=k}} \binom{k}{s_1,\ldots,s_k} \left(-\frac{\psi_1}{1}\right)^{s_1} \ldots \left(-\frac{\psi_k}{k}\right)^{s_k} \tag{14}$$

Let $w = y_{s_1}\ldots y_{s_r} \in Y^*$. The quasi-symmetric function F_w, of depth $r = |w|$ and of degree (or weight) $s_1 + \ldots + s_r$, is defined by

$$F_w(\underline{t}) = \sum_{n_1>\ldots>n_r>0} t_{n_1}^{s_1}\ldots t_{n_r}^{s_r}. \tag{15}$$

In particular, $F_{y_1^k} = \lambda_k$ and $F_{y_k} = \psi_k$. As a consequence, the functions $\{F_{y_1^k}\}_{k\geq 0}$ are linearly independent and integrating differential equation (13) shows that functions $F_{y_1^k}$ and F_{y_k} are linked by the formula

$$\sum_{k\geq 0} F_{y_1^k} z^k = \exp\left[-\sum_{k\geq 1} F_{y_k}\frac{(-z)^k}{k}\right]. \tag{16}$$

Remarkable identity (14) can be then seen as :

$$k!y_1^k = (-1)^k \sum_{\substack{s_1,\ldots,s_k>0\\ s_1+\ldots+ks_k=k}} \binom{k}{s_1,\ldots,s_k} \frac{(-y_1)^{\text{⧢} s_1}}{1^{s_1}} \text{⧢} \ldots \text{⧢} \frac{(-y_k)^{\text{⧢} s_k}}{k^{s_k}} \tag{17}$$

Every $\mathrm{H}_w(N)$ can be obtained by specializing variables $\{t_i\}_{N\geq i\geq 1}$ at $t_i = 1/i$ and, for $i > N, t_i = 0$ in the quasi-symmetric function F_w [5]. In the same way, when $w \in Y^* \setminus y_1Y^*$, the convergent polyzêta $\zeta(w)$ can be obtained by specializing variables $\{t_i\}_{i\geq 1}$ at $t_i = 1/i$ in F_w [5]. The notation F_w is extended by linearity to all polynomials over Y.

If u (resp. v) is a word in Y^*, of length r and of weight p (resp. of length s and of weight q), $F_{u \text{⧢} v}$ is a quasi-symmetric function of depth $r+s$ and of weight $p+q$, and we have $F_{u \text{⧢} v} = F_u\, F_v$.

In consequence, $\mathrm{H}_{u \sqcup\!\sqcup v} = \mathrm{H}_u \, \mathrm{H}_v$ [5]. In the same way, when $u, v \in Y^* \backslash y_1 Y^*$, we also have $\zeta(u \sqcup\!\sqcup v) = \zeta(u) \, \zeta(v)$ [5].

Let us consider the noncommutative generating series $\mathrm{H}(N)$ of $\{\mathrm{H}_w(N)\}_{w \in Y^*}$ [14],

$$\mathrm{H}(N) = \sum_{w \in Y^*} \mathrm{H}_w(N) \; w = \prod_{1}^{l=N} \Big(1 + \sum_{i>0} \frac{y_i}{l^i}\Big). \tag{18}$$

since it verifies the difference equation

$$\mathrm{H}(N) = \Big(1 + \sum_{i>0} \frac{y_i}{N^i}\Big) \mathrm{H}(N-1), \quad \text{with} \quad \mathrm{H}(0) = 1. \tag{19}$$

2.1.2 Polylogarithms and polyzêtas

Let us denote by $\mathcal{C}$ the algebra $\mathbb{C}[z, 1/z, 1/(1-z)]$ of polynomial functions in $z, 1/z$ and $1/(1-z)$. We define two differential forms $\omega_0(z) = dz/z$ and $\omega_1(z) = dz/(1-z)$.

Let $w = x_0^{s_1-1} x_1 \ldots x_0^{s_r-1} x_1 \in X^* x_1$. One can check that the polylogarithm Li_w is also the value of the iterated integral over ω_0, ω_1 and along the integration path $0 \rightsquigarrow z$:

$$\mathrm{Li}_w = \int_{0 \rightsquigarrow z} \omega_0^{s_1-1} \omega_1 \ldots \omega_0^{s_r-1} \omega_1. \tag{20}$$

This provides an analytic continuation of the Li_w over the universal covering $\widetilde{\mathbb{C} - \{0,1\}}$ of $\mathbb{C}$ without points 0 and 1. We extend the definition of polylogarithms over X^* putting

$$\mathrm{Li}_{x_0^k}(z) = \log^k z / k!, \quad \text{for} \quad k \in \mathbb{N}. \tag{21}$$

Let $\mathrm{LI}_{\mathcal{C}} = (\mathcal{C}\{\mathrm{Li}_w\}_{w \in X^*}, .)$ be the smallest $\mathcal{C}$-algebra containing $\mathcal{C}$ and stable by differentiation and integration over ω_0, ω_1. It can be identified with the $\mathcal{C}$-module generated by polylogarithms. Thus, the polylogarithms are $\mathcal{C}$-linearly independent [6]. Hence, $(\mathcal{C}\{\mathrm{Li}_w\}_{w \in X^*}, .)$ is identified with the polynomial algebra $(\mathcal{C}\{\mathrm{P}_l\}_{l \in \mathcal{L}ynX}, .)$ [6].

The noncommutative generating series $\mathrm{L}(z)=\sum_{w\in X^*}\mathrm{Li}_w(z)\,w$ satisfies Drinfel'd differential equation [7, 8]

$$d\mathrm{L}=(x_0\omega_0+x_1\omega_1)\mathrm{L},\ \text{with the condition}\\ \mathrm{L}(\varepsilon)=e^{x_0\log\varepsilon}+O(\sqrt{\varepsilon})\ \text{for}\ \varepsilon\to 0^+. \tag{22}$$

This enables to prove that L is the exponential of a Lie series [6]. So, applying a Ree theorem, it verifies Friedrichs criterion [6], *i..e* $\mathrm{Li}_{u\,\sqcup\!\sqcup\,v}=\mathrm{Li}_u\ \mathrm{Li}_v$ for $u,v\in X^*$. In particular, when $u,v\in x_0X^*x_1$, we also have $\zeta(u\sqcup\!\sqcup v)=\zeta(u)\,\zeta(v)$. From the factorization of monoid by Lyndon words, we obtain the factorization of the series L [6] :

$$\mathrm{L}(z)=e^{x_1\log\frac{1}{1-z}}\left[\prod_{l\in\mathcal{L}ynX\setminus\{x_0,x_1\}}^{\searrow}e^{\mathrm{Li}_{S_l}(z)[l]}\right]e^{x_0\log z}. \tag{23}$$

For all $l\in\mathcal{L}ynX\setminus\{x_0,x_1\}$, we have $S_l\in x_0X^*x_1$. So, let us put [6]

$$\mathrm{L}_{\mathrm{reg}}=\prod_{l\in\mathcal{L}ynX\setminus\{x_0,x_1\}}^{\searrow}e^{\mathrm{Li}_{S_l}[l]}\quad\text{and}\quad Z=\mathrm{L}_{\mathrm{reg}}(1). \tag{24}$$

Let σ be the monoid endomorphism verifying $\sigma(x_0)=-x_1,\sigma(x_1)=-x_0$, we also get [9]

$$\mathrm{L}(z)=\sigma[\mathrm{L}(1-z)]Z=e^{x_0\log z}\sigma[\mathrm{L}_{\mathrm{reg}}(1-z)]e^{-x_1\log(1-z)}Z. \tag{25}$$

In consequence, from (23) and (25), we get respectively

$$\mathrm{L}(z)\ \widetilde{_{z\to0}}\ \exp(x_0\log z)\quad\text{and}\quad \mathrm{L}(z)\ \widetilde{_{z\to1}}\ \exp\left(x_1\log\frac{1}{1-z}\right)Z. \tag{26}$$

Let $\pi_Y:\mathrm{LI}_{\mathcal{C}}\langle\langle X\rangle\rangle\to\mathrm{LI}_{\mathcal{C}}\langle\langle Y\rangle\rangle$ a projector s.t., for $f\in\mathrm{LI}_{\mathcal{C}},w\in X^*,\pi_Y(f\ wx_0)=0$. Then

$$\Lambda(z)=\pi_Y\mathrm{L}(z)\ \widetilde{_{z\to1}}\ \exp\left(y_1\log\frac{1}{1-z}\right)\pi_YZ. \tag{27}$$

Definition 1 ([1]). *Let* $\zeta_{\sqcup\!\sqcup} : (C\langle\langle X\rangle\rangle, \sqcup\!\sqcup) \to (\mathbb{C}, .)$ *be the algebra morphism (*i.e. *for* $u, v \in X^*, \zeta_{\sqcup\!\sqcup}(u \sqcup\!\sqcup v) = \zeta_{\sqcup\!\sqcup}(u)\zeta_{\sqcup\!\sqcup}(v)$*) verifying for all convergent word* $w \in x_0X^*x_1, \zeta_{\sqcup\!\sqcup}(w) = \zeta(w)$*, and such that* $\zeta_{\sqcup\!\sqcup}(x_0) = \zeta_{\sqcup\!\sqcup}(x_1) = 0$.

Then, the noncommutative generating series $Z_{\sqcup\!\sqcup} = \sum_{w\in X^*} \zeta_{\sqcup\!\sqcup}(w)\ w$ verifies $Z_{\sqcup\!\sqcup} = Z$ [1]. In consequence, $Z_{\sqcup\!\sqcup}$ is the unique Lie exponential verifying $\langle Z_{\sqcup\!\sqcup}|x_0\rangle = \langle Z_{\sqcup\!\sqcup}|x_1\rangle = 0$ and $\langle Z_{\sqcup\!\sqcup}|w\rangle = \zeta(w)$, for any $w \in x_0X^*x_1$. Its logarithm is given by $\log Z_{\sqcup\!\sqcup} = \sum_{w\in X^*} \zeta_{\sqcup\!\sqcup}(w)\ \pi_1(w)$, where $\pi_1(w)$ is the Lie polynomial [3]

$$\pi_1(w) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\cdots,u_k\in X^*\setminus\{\epsilon\}} \langle w|u_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup u_k\rangle\ u_1\cdots u_k. \tag{28}$$

The series Z shall be understood then as Drinfel'd associator Φ_{KZ} [7, 8] verifying duality, pentagonal and hexagonal relations. We also can obtain the expression of this associator given by Lê and Murakami [11] thanks to the following expansion [10]

$$Z = \sum_{k\geq 0} \sum_{l_1,\cdots,l_k\geq 0} \zeta_{\sqcup\!\sqcup}(x_1x_0^{l_1} \circ \cdots \circ x_1x_0^{l_k}) \prod_{i=1}^{k} \mathrm{ad}_{x_0}^{l_i} x_1, \tag{29}$$

where $\mathrm{ad}_{x_0}^{l} x_1$ stands for the iterated Lie bracket $[x_0, [\ldots, [x_0, x_1]\ldots]$, for $l > 0$ and $\mathrm{ad}_{x_0}^{0} x_1 = x_1$, the operation $\circ$ being defined as $x_1x_0^l \circ P = x_1(x_0^l \sqcup\!\sqcup P)$, for any $P \in \mathbb{C}\langle X\rangle$.

2.1.3 Ordinary generating series of harmonic sums

The functions $\{\mathrm{Li}_w\}_{w\in X^*}$ are $\mathcal{C}$-linearly independent. Thus, the functions $\{\mathrm{P}_w\}_{w\in Y^*}$ are also $\mathbb{C}$-linearly independent. In consequence, functions $\{\mathrm{H}_w\}_{w\in Y^*}$ are also $\mathbb{C}$-linearly independent [13, 12]. So,

Proposition 1 ([14]). *Extended by linearity, the application* $\mathrm{P} : u \mapsto \mathrm{P}_u$ *is an isomorphism from polynomial algebra* $(\mathbb{C}\langle Y\rangle, \uplus)$ *over Hadamard algebra* $(\mathbb{C}\{\mathrm{P}_w\}_{w\in Y^*}, \odot)$*. Moreover, the application* $\mathrm{H} : u \mapsto \mathrm{H}_u = \{\mathrm{H}_u(N)\}_{N\geq 0}$ *is an isomorphism from* $(\mathbb{C}\langle Y\rangle, \uplus)$ *over the algebra* $(\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, .)$.

Proof : Indeed, on the first hand, $\ker \mathrm{P} = \{0\}$ and $\ker \mathrm{H} = \{0\}$, and on the other hand, P is a morphism for Hadamard product (it inherits of H for the harmonic product) :

$$\begin{aligned} \mathrm{P}_u(z) \odot \mathrm{P}_v(z) &= \sum_{N\geq 0} \mathrm{H}_u(N)\mathrm{H}_v(N) z^N \\ &= \sum_{N\geq 0} \mathrm{H}_{u \uplus v}(N) z^N \\ &= \mathrm{P}_{u \uplus v}(z). \end{aligned}$$

Studying the equivalence between action of $\{(1-z)^l\}_{l\in\mathbb{Z}}$ over $\{\mathrm{P}_w(z)\}_{w\in Y^*}$ and this of $\{N^k\}_{k\in\mathbb{Z}}$ over $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ [12], we have

Theorem 1. *The Hadamard $\mathcal{C}$-algebra of $\{\mathrm{P}_w\}_{w\in Y^*}$ can be identified with this of $\{\mathrm{P}_l\}_{l\in\mathcal{L}ynY}$. Identically, the algebra of harmonic sums $\{\mathrm{H}_w\}_{w\in Y^*}$ with polynomial coefficients can be identified with this of $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$.*

As for polylogarithms, we extend the definition of P_w putting $\mathrm{P}_w(z) = (1-z)^{-1}\mathrm{Li}_w(z)$, for any $w \in X^*$. The noncommutative generating series of $\{\mathrm{P}_w\}_{w\in X^*}$ is defined by

$$\mathrm{P}(z) = \sum_{w\in X^*} \mathrm{P}_w(z)\, w = \frac{\mathrm{L}(z)}{1-z}. \tag{30}$$

In consequence, by (23), we have

$$\mathrm{P}(z) = e^{-(x_1+1)\log(1-z)}\mathrm{L}_{\mathrm{reg}}(z) e^{x_0 \log z}. \tag{31}$$

Lemma 1. *Let* $\mathrm{Mono}(z) = e^{-(x_1+1)\log(1-z)}$. *Then*

$$\mathrm{Mono} = \sum_{k\geq 0} \mathrm{P}_{y_1^k}\, y_1^k, \quad \textit{and} \quad \mathrm{Mono}^{-1} = \sum_{k\geq 0} \mathrm{P}_{y_1^k}\, (-y_1)^k.$$

Since the coefficient of z^N in the Taylor expansion of $\mathrm{P}_{y_1^k}$ is $\mathrm{H}_{y_1^k}(N)$ then

Lemma 2. *Let* $\mathrm{Const} = \sum_{k\geq 0} \mathrm{H}_{y_1^k}\, y_1^k$. *Then*

$$\mathrm{Const} = \exp\left[-\sum_{k\geq 1} \mathrm{H}_{y_k} \frac{(-y_1)^k}{k}\right] \quad \textit{and} \quad \mathrm{Const}^{-1} = \exp\left[\sum_{k\geq 1} \mathrm{H}_{y_k} \frac{(-y_1)^k}{k}\right].$$

Proof : This is a consequence of Formula (16).

Proposition 2. *For $k>0, \mathrm{H}_{y_1^k}$ is polynomial in $\{\mathrm{H}_{y_r}\}_{1\le r\le k}$ (which are algebraically independent), and*

$$\mathrm{H}_{y_1^k}=\sum_{\substack{s_1,\ldots,s_k>0\\ s_1+\ldots+ks_k=k}}\frac{(-1)^k}{s_1!\ldots s_k!}\left(-\frac{\mathrm{H}_{y_1}}{1}\right)^{s_1}\ldots\left(-\frac{\mathrm{H}_{y_k}}{k}\right)^{s_k}.$$

Proof : From Identity (17), and applying the isomorphism H on the set of Lyndon words $\{y_r\}_{1\le r\le k}$, we get the expected result.

Example 1. $\mathrm{H}_{y_1^2}=(\mathrm{H}_{y_1}^2-\mathrm{H}_{y_2})/2$, $\mathrm{H}_{y_1^3}=(\mathrm{H}_{y_1}^3-3\mathrm{H}_{y_2}\mathrm{H}_{y_1}+2\mathrm{H}_{y_3})/6$.

Proposition 3 ([14]). *Let σ be the morphism verifying $\sigma(x_0)=-x_1,\sigma(x_1)=-x_0$.*

$$\mathrm{P}(z)=e^{x_0\log z}\left[\prod_{l\in\mathcal{L}ynX,\setminus\{x_0,x_1\}}^{\searrow}e^{\mathrm{Li}_{S_l}(1-z)\sigma([l])}\right]\mathrm{Mono}(z)Z,$$

Proof : On the first hand, from (31) and on the other hand, from (25), we get $\mathrm{P}(z)=e^{x_0\log z}\sigma[\mathrm{L}_{\mathrm{reg}}(1-z)]e^{-(x_1+1)\log(1-z)}Z$. Using the expressions of $\mathrm{L}_{\mathrm{reg}}(1-z)$ and of $\mathrm{Mono}(z)$, we get the expected results.

2.2 Asymptotic expansion

2.2.1 Results à l'Abel for generating series

Proposition 4. $\mathrm{P}(z)\underset{z\to0}{\widetilde{\ }}e^{x_0\log z}$ *and* $\mathrm{P}(z)\underset{z\to1}{\widetilde{\ }}\mathrm{Mono}(z)Z$.

Proof : From $\mathrm{P}(z)=e^{-(x_1+1)\log(1-z)}\mathrm{L}_{\mathrm{reg}}(z)e^{x_0\log z}$, we can deduce the behaviour of $\mathrm{P}(z)$ around 0. From Formula (25), we get the behaviour of $\mathrm{P}(z)$ around 1.

Corollary 1. *Let $\Pi(z)=\pi_Y\mathrm{P}(z)=\sum_{w\in Y^*}\mathrm{P}_w(z)\ w$. Then $\Pi(z)\underset{z\to1}{\widetilde{\ }}\mathrm{Mono}(z)\pi_YZ$.*

From this, we extract, taking care of Lemma 1, Taylor coefficients of P_w, and we get

Corollary 2. $\mathrm{H}(N) \underset{N\to\infty}{\widetilde{\quad}} \mathrm{Const}(N)\pi_Y Z.$

Theorem 2.

$$\lim_{z\to1} \exp\left(y_1 \log \frac{1}{1-z}\right)\Lambda(z) = \lim_{N\to\infty} \exp\left(\sum_{k\geq1} \mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right)\mathrm{H}(N) = \pi_Y Z.$$

Proof : This is a consequence of Formula (27), of Lemma 2 and of Corollary 2.

From Proposition 3, we deduce

$$\mathrm{P}(z) = e^{x_0 \log z}\left[\prod_{\substack{l\in\mathcal{L}ynX,\\ l\neq x_0,x_1}}^{\searrow} z\left(\sum_{k\geq0} \mathrm{P}_{S_l^{\sqcup\!\sqcup k}}(1-z)\frac{(\sigma([l]))^k}{k!}\right)\right]\mathrm{Mono}(z)Z. \tag{32}$$

Hence, the knowledge of Taylor expansion around 0 of $\{\mathrm{P}_w(1-z)\}_{w\in X^*}$ gives

Theorem 3 ([12]). *For all $g \in \mathcal{C}\{\mathrm{P}_w\}_{w\in Y^*}$, there exist algorithmically computable coefficients $c_j \in \mathbb{C}$, $\alpha_j \in \mathbb{Z}$ and $\beta_j \in \mathbb{N}$ such that*

$$g(z) \sim \sum_{j=0}^{+\infty} c_j(1-z)^{\alpha_j}\log^{\beta_j}(1-z) \quad \textit{for} \quad z\to1.$$

In consequence, there exist algorithmically computable coefficients $b_i \in \mathbb{C}$, $\eta_i \in \mathbb{Z}$ and $\kappa_i \in \mathbb{N}$ such that

$$[z^n]g(z) \sim \sum_{i=0}^{+\infty} b_i n^{\eta_i}\log^{\kappa_i}(n) \quad \textit{for} \quad n\to\infty.$$

Corollary 3 ([12]). *Let $\mathcal{Z}$ the $\mathbb{Q}$-algebra generated by convergent polyzêtas and $\mathcal{Z}'$ the $\mathbb{Q}[\gamma]$-algebra generated by $\mathcal{Z}$. Then there exist algorithmically computable coefficients $c_j \in \mathcal{Z}$, $\alpha_j \in \mathbb{Z}$ and $\beta_j \in \mathbb{N}$ such that*

$$\forall w \in Y^*, \mathrm{P}_w(z) \sim \sum_{j=0}^{+\infty} c_j(1-z)^{\alpha_j}\log^{\beta_j}(1-z) \quad \textit{for} \quad z\to1.$$

In consequence, there exist algorithmically computable coefficients $b_i \in \mathcal{Z}', \kappa_i \in \mathbb{N}$ *and* $\eta_i \in \mathbb{Z}$ *such that*

$$\forall w \in Y^*, \mathrm{H}_w(N) \sim \sum_{i=0}^{+\infty} b_i N^{\eta_i} \log^{\kappa_i}(N) \quad \textit{for} \quad N \to +\infty.$$

2.2.2 Generalized Euler constants associated to divergent polyzêtas

Definition 2. *Let* $\zeta_{⧢} : (\mathbb{C}\langle Y\rangle, ⧢) \to (\mathbb{C}, .)$ *the algebra morphism (*i.e. *for all convergent word* $u, v \in Y^*, \zeta_{⧢}(u ⧢ v) = \zeta_{⧢}(u)\zeta_{⧢}(v)$*) verifying for* $w \in Y^* \setminus y_1 Y^*$, $\zeta_{⧢}(w) = \zeta(w)$ *and such that* $\zeta_{⧢}(y_1) = \gamma$.

Proposition 5.

$$\zeta_{⧢}(y_1^k) = \sum_{\substack{s_1,\ldots,s_k>0 \\ s_1+\ldots+ks_k=k}} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k}.$$

Proof : By (17) and applying the (surjective) morphism $\zeta_{⧢}$, we get the expected result.

In consequence,

Theorem 4. *For* $k > 0$*, the constant* $\zeta_{⧢}(y_1^k)$ *associated to divergent polyzêta* $\zeta(y_1^k)$ *is a polynomial of degree* k *in* γ *with coefficients in* $\mathbb{Q}[\zeta(2), \zeta(2i+1)]_{0<i\leq(k-1)/2}$*. Moreover, for* $l = 0, .., k$*, the coefficient of* γ^l *is of weight* $k - l$*.*

Example 2. $\zeta_{⧢}(y_1^2) = [\gamma^2 - \zeta(2)]/2, \zeta_{⧢}(y_1^3) = [\gamma^3 - 3\zeta(2)\gamma + 2\zeta(3)]/6$ *and* $\zeta_{⧢}(y_1^4) = [80\zeta(3)\gamma - 60\zeta(2)\gamma^2 + 6\zeta(2)^2 + 10\gamma^4]/240.$

Let us consider (exponential) partial Bell polynomials partiels in the variables $\{t_l\}_{l\geq1}$, $b_{n,k}(t_1, \ldots, t_{n-k+1})$, defined by the exponential generating series :

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} b_{n,k}(t_1, \ldots, t_{n-k+1})\frac{v^n u^k}{n!} = \exp\left(u\sum_{l=1}^{\infty} t_l \frac{v^l}{l!}\right). \tag{33}$$

In particular, we have

Lemma 3. *Let $t_m = (-1)^m (m-1)! \zeta_{\sqcup\!\sqcup}(m)$, for $m \geq 1$, then*

$$\exp\left[\sum_{k\geq 1} \zeta_{\sqcup\!\sqcup}(k) \frac{(-y_1)^k}{k}\right] = 1 + \sum_{n\geq 1}\left[\sum_{k=1}^{n} b_{n,k}(\gamma, \zeta(2), 2\zeta(3), \ldots)\right] \frac{(-y_1)^n}{n!}.$$

Let us build the noncommutative generating series of $\zeta_{\sqcup\!\sqcup}(w)$ and let us take the constant part of the two members of $\mathrm{H}(N) \underset{N\to\infty}{\widetilde{}} \mathrm{Const}(N)\pi_Y Z$, we have

Proposition 6. *Let $Z_{\sqcup\!\sqcup}$ be the noncommutative generating series of the constants $\zeta_{\sqcup\!\sqcup}(w)$, i.e. $Z_{\sqcup\!\sqcup} = \sum_{w\in Y^*} \zeta_{\sqcup\!\sqcup}(w)\, w$. Then*

$$Z_{\sqcup\!\sqcup} = \left[1 + \sum_{n\geq 1}\left(\sum_{k=1}^{n} b_{n,k}(\gamma, \zeta(2), 2\zeta(3), \ldots)\right) \frac{(-y_1)^n}{n!}\right] \pi_Y Z.$$

Identifying coefficients of $y_1^k w$ in each member leads to

Corollary 4. *For all $w \in Y^* \setminus y_1 Y^*$ and $k \geq 0$, we have*

$$\zeta_{\sqcup\!\sqcup}(y_1^k w) = \sum_{i=1}^{k} \frac{\zeta_{\sqcup\!\sqcup}(y_1^{k-i} w)}{i!}\left[(-1)^i \sum_{j=1}^{i} b_{i,j}(\gamma, \zeta(2), 2\zeta(3), \ldots)\right].$$

Theorem 5. *In consequence, for $w \in Y^* \setminus y_1 Y^*, k \geq 0$, the constant $\zeta_{\sqcup\!\sqcup}(y_1^k w)$ associated to $\zeta(y_1^k w)$ is a polynomial of de degree k in γ and with coefficients in $\mathcal{Z}$. Moreover, for $l = 0, .., k$, the coefficient of γ^l is of weight $|w| + k - l$.*

Corollary 5. *For $s > 1$, the constant $\zeta_{\sqcup\!\sqcup}(1, s)$ associated to $\zeta(1, s)$ is linear in γ and with coefficients in $\mathbb{Q}[\zeta(2), \zeta(2i+1)]_{0<i\leq(s-1)/2}$.*

Example 3. $\gamma = \dfrac{\zeta_{\sqcup\!\sqcup}(1,2) + 2\zeta(3)}{\zeta(2)} = \dfrac{\zeta_{\sqcup\!\sqcup}(1,3) + \frac{1}{2}\zeta(2)^2}{\zeta(3)} = \dfrac{\zeta_{\sqcup\!\sqcup}(1,4) + 3\zeta(5) - \zeta(2)\zeta(3)}{\frac{2}{5}\zeta(2)^2}.$

In other words, if we give to γ the weight[2] 1, then the constant $\zeta_{⧢}(y_1^k w)$ associated to $\zeta(y_1^k w)$ would be an homogeneous polynomial of weight $|w|+k$.

Example 4.

$$\begin{aligned}
\zeta_{⧢}(1,7) = &\quad \zeta(7)\gamma + \zeta(3)\zeta(5) - \tfrac{54}{175}\zeta(2)^4,\\
\zeta_{⧢}(1,1,6) = &\ \tfrac{4}{35}\zeta(2)^3\gamma^2 + \Big(\zeta(2)\zeta(5) + \tfrac{2}{5}\zeta(3)\zeta(2)^2 - 4\zeta(7)\Big)\gamma\\
&+\zeta(6,2) + \tfrac{19}{35}\zeta(2)^4 + \tfrac{1}{2}\zeta(2)\zeta(3)^2 - 4\zeta(3)\zeta(5),\\
\zeta_{⧢}(1,1,1,5) = &\ \tfrac{3}{4}\zeta(6,2) - \tfrac{14}{3}\zeta(3)\zeta(5) + \tfrac{3}{4}\zeta(2)\zeta(3)^2 + \tfrac{809}{1400}\zeta(2)^4\\
&-\Big(2\zeta(7) - \tfrac{3}{2}\zeta(2)\zeta(5) + \tfrac{1}{10}\zeta(3)\zeta(2)^2\Big)\gamma\\
&+\Big(\tfrac{1}{4}\zeta(3)^2 - \tfrac{1}{5}\zeta(2)^3\Big)\gamma^2 + \tfrac{1}{6}\zeta(5)\gamma^3.
\end{aligned}$$

2.2.3 Commutative generating series of polyzêtas

In Proposition 6, we explained how to extract the constant part of a non-commutative generating series of polyzêtas. Let us have a look now at following commutative generating series and corresponding to Ecalle's *Zig*-moulds [15] :

$$\begin{aligned}
\mathcal{Z}(t_1,\cdots,t_r) &= \sum_{s_1,\cdots,s_r>0} \zeta(s_1,\cdots,s_r)\, t_1^{s_1-1}\cdots t_r^{s_r-1} \\
&= \sum_{n_1>\cdots>n_r>0} \frac{1}{(n_1-t_1)\cdots(n_r-t_r)}.
\end{aligned} \tag{34}$$

These commutative generating series can be encoded by series $\{S_j\}_{j=1,..,r}$ (or their projection over the alphabet Y) of the form [1]

[2] In the theory of periods, γ is conjectured to be non-period and so would be transcendent.

$$
\begin{aligned}
S_j =& (t_j x_0)^* x_1 \ldots (t_r x_0)^* x_1 && (35)\\
=& \sum_{s_j,\ldots,s_r>0} x_0^{s_j-1} x_1 \ldots x_0^{s_r-1} x_1 t_j^{s_j-1} \ldots t_r^{s_r-1},\\
\pi_Y S_j =& \sum_{s_j,\ldots,s_r>0} y_{s_j} \ldots y_{s_r} t_j^{s_j-1} \ldots t_r^{s_r-1} && (36)\\
=& \Big(\sum_{s_j\geq 1} y_{s_j} t_j^{s_j-1}\Big) \ldots \Big(\sum_{s_r\geq 1} y_{s_r} t_r^{s_r-1}\Big).
\end{aligned}
$$

Moreover, let $S_{r+1} = 1$. The series $\mathcal{Z}(t_1, \cdots, t_r)$ contain divergent terms of which we are looking for the constant part. We start from the following identity [1] due to convolution theorem [16]

$$
\begin{aligned}
S_1 &= x_1^r + \sum_{j=1}^{r} t_j \sum_{i=0}^{j-1} x_1^i ⧢ x_0[(-x_1)^{j-1-i} ⧢ S_j], && (37)\\
\Rightarrow \quad \pi_Y S_1 &= y_1^r + \sum_{j=1}^{r} t_j \Big(\sum_{s_j\geq 2} y_1^{j-1} y_{s_j} t_j^{s_j-1}\Big) \pi_Y S_{j+1}. && (38)
\end{aligned}
$$

Proposition 7. *In consequence,*

$$
\zeta_{⧢}(S_1) = \sum_{j=1}^{r} (-1)^{j-1} t_j \, \zeta[x_0(x_1^{j-1} ⧢ S_j)]. \qquad (39)
$$

$$
\begin{aligned}
\zeta_{\text{⧢}}(\pi_Y S_1) =& (-1)^r \sum_{\substack{s_1,\ldots,s_r>0\\ s_1+\ldots+rs_r=r}} \frac{(-\gamma)^{s_1}}{s_1! \ldots s_r!} \prod_{j=2}^{r} \Big(-\frac{\zeta(j)}{j}\Big)^{s_j} && (40)\\
&+ \sum_{j=1}^{r} \sum_{s_j\geq 2} t_j t_j^{s_j-1} \, \zeta_{\text{⧢}}(y_1^{j-1} y_{s_j} \pi_Y S_{j+1}).
\end{aligned}
$$

In the following part, iterated integrals associated with words $w \in X^*$, along the path $0 \rightsquigarrow z$ and over differential forms ω_0, ω_1, will be denoted, as in [1], by $\alpha_0^z(w)$.

Example 5. *Since* $(tx_0)^*x_1 = x_1 + tx_0(tx_0)^*x_1$,

$$\alpha_0^z[(tx_0)^*x_1] = \alpha_0^z(x_1) + t\int_0^z \left(\frac{z}{s}\right)^t \mathrm{Li}_1(s)\frac{ds}{s} = \alpha_0^z(x_1) + t\sum_{n\geq 1}\frac{z^n}{n(n-t)}.$$

we get

$$\begin{aligned}\sum_{s\geq 1}\zeta_{\sqcup\!\sqcup}(s)t^{s-1} &= \sum_{s\geq 2}\zeta(s)t^{s-1} &= \sum_{n\geq 1}\left[\frac{1}{n-t}-\frac{1}{n}\right],\\ \sum_{s\geq 1}\zeta_{\uplus}(s)t^{s-1} &= \gamma + \sum_{s\geq 2}\zeta(s)t^{s-1} &= \gamma + \sum_{n\geq 1}\left[\frac{1}{n-t}-\frac{1}{n}\right].\end{aligned}$$

Example 6. *Identity (37) gives*

$$\begin{aligned}S =& x_1^2 + t_1\, x_0(t_1x_0)^*x_1(t_2x_0)^*x_1 + t_2\, x_0\,(-x_1 \sqcup\!\sqcup (t_2x_0)^*x_1)\\ &+ t_2\, x_1 \sqcup\!\sqcup x_0(t_2x_0)^*x_1.\end{aligned} \tag{41}$$

In the second member of the previous expression,

- *we have* $\zeta_{\sqcup\!\sqcup}(x_1^2) = 0$,
- *the first noncommutative rational series encodes, by convolution theorem [16], the following convergent integral*

$$\begin{aligned}\alpha_0^1[(t_1x_0)^*x_0x_1(t_2x_0)^*x_1] &= \int_0^1 \left(\frac{1}{s}\right)^{t_1}\frac{ds}{s}\int_0^s \frac{dr}{1-r}\sum_{n\geq 1}\frac{r^n}{n-t_2}\\ &= \sum_{n,m\geq 1}\frac{1}{(n+m)(n+m-t_1)(n-t_2)}\\ &= \sum_{n_1>n_2\geq 1}\frac{1}{n_1(n_1-t_1)(n_2-t_2)},\end{aligned}$$

- *the following rational series encodes*

$$\begin{aligned}\alpha_0^1[x_0\,(-x_1 ш (t_2x_0)^*x_1)] &= -\int_0^1 \frac{ds}{s}\sum_{m\geq 1}\frac{s^m}{m}\sum_{n\geq 1}\frac{s^n}{n-t_2}\\ &= -\sum_{n,m\geq 1}\frac{1}{(n-t_2)(n+m)n}\\ &= -\sum_{n_1>n_2\geq 1}\frac{1}{n_1(n_1-n_2)(n_2-t_2)},\end{aligned}$$

- *the last noncommutative rational series corresponds to (but it has to be shuffled with x_1 for which $\zeta_ш(x_1)=0$)*

$$\alpha_0^1[x_0(t_2x_0)^*x_1] = \int_0^1 \frac{ds}{s}\sum_{n\geq 1}\frac{s^n}{n-t_2} = \sum_{n\geq 1}\frac{1}{n(n-t_2)}.$$

Thus,

$$\begin{aligned}\sum_{s,r\geq 1}\zeta_ш(s,r)t_1^{s-1}t_2^{r-1} = &\sum_{n_1>n_2\geq 1}\frac{t_1}{n_1(n_1-t_1)(n_2-t_2)}\\ &-\sum_{n_1>n_2\geq 1}\frac{t_2}{n_1(n_1-n_2)(n_2-t_2)}.\end{aligned} \tag{42}$$

Projecting S over alphabet Y, we get successively, since $y_1y_s = y_1 ш y_s - y_{s+1} - y_sy_1$

$$
\begin{aligned}
\pi_Y S =& y_1^2 + \sum_{s,r\geq 2} y_s y_r t_1^{s-1} t_2^{r-1} + \sum_{s\geq 2} y_1 y_s t_2^{s-1} + \sum_{s\geq 2} y_s y_1 t_1^{s-1} \\
=& y_1^2 + \sum_{s,r\geq 2} y_s y_r t_1^{s-1} t_2^{r-1} + y_1 \,ш \sum_{s\geq 2} y_s t_2^{s-1} - \frac{1}{t_2}\left[y_1 + \sum_{s\geq 2} y_s t_2^{s-1} \right] \\
&+ \sum_{s\geq 2} y_s y_1 [t_1^{s-1} - t_2^{s-1}] \\
=& y_1^2 - y_1 t_2^{-1} + (y_1 - t_2^{-1}) \sum_{s\geq 2} y_s t_2^{s-1} + \sum_{s,r\geq 2} y_s y_r t_1^{s-1} t_2^{r-1} \\
&+ \sum_{s\geq 2} y_s y_1 [t_1^{s-1} - t_2^{s-1}].
\end{aligned}
$$

In consequence,

$$
\begin{aligned}
\sum_{s,r\geq 1} \zeta_{ш}(s,r) t_1^{s-1} t_2^{r-1} &= \frac{\gamma^2 - \zeta(2)}{2} - \frac{\gamma}{t_2} + (\gamma - t_2^{-1}) \sum_{n\geq 1} \left[\frac{1}{n - t_2} - \frac{1}{n} \right] \\
&+ \sum_{n_1 > n_2 \geq 1} \left[\frac{1}{n_1 - t_1} - \frac{1}{n_1} \right] \left[\frac{1}{n_2 - t_2} - \frac{1}{n_2} \right] \\
&+ \sum_{s\geq 2} \zeta(s,1) [t_1^{s-1} - t_2^{s-1}].
\end{aligned}
$$

The last sum can be encoded by $x_0(t_1 x_0)^* x_1^2 - x_0(t_2 x_0)^* x_1^2$ *and can be obtained from*

$$
\alpha_0^1[x_0(t_i x_0)^* x_1^2] = \int_0^1 \frac{ds}{s} \sum_{n_1 > n_2 \geq 1} \frac{s^{n_1}}{(n_1 - t_i) n_2} = \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1 (n_1 - t_i) n_2}.
$$

Appendix : application to polysystems [14]

Let $q_1, \ldots, q_n$ be commutative indeterminates over $\mathbb{C}$. We denote $Q = \{q_1, \ldots, q_n\}$. The algebra of formal power series (resp. polynomials) over Q with coefficients in $\mathbb{C}$ is denoted by $\mathbb{C}[[Q]]$ (resp. $\mathbb{C}[Q]$). An element of $\mathbb{C}[[Q]]$ is an infinite sum $f = \sum_{i_1,\ldots,i_n \geq 0} f_{i_1,\ldots,i_n} q_1^{i_1} \ldots q_n^{i_n}$.

Definition 3. *Let $f \in \mathbb{C}[[Q]]$. We set*

$$\begin{aligned} E(f) &= \{\rho \in \mathbb{R}_+^n : \exists C_{f\rho} \in \mathbb{R}_+ \text{ such that for all } i_1, \ldots, i_n \geq 0, \\ &\quad |f_{i_1,\ldots,i_n}|\rho_1^{i_1} \ldots \rho_n^{i_n} \leq C_{f\rho}\} \\ \check{E}(f) &: \text{ interior of } E(f) \text{ in } \mathbb{R}^n. \\ \mathrm{CV}(f) &= \text{convergence domain of } f = \{q \in \mathbb{C}^n : (|q_1|, \ldots, |q_n|) \in \check{E}(f)\}. \end{aligned}$$

The power series f is to be said convergent *if $\mathrm{CV}(f) \neq \emptyset$. Let $\mathcal{U}$ be an open of $\mathbb{C}^n$ and let $q \in \mathbb{C}^n$. The power series f is to be said convergent on q (resp. over $\mathcal{U}$) if $q \in \mathrm{CV}(f)$ (resp. $\mathcal{U} \subset \mathrm{CV}(f)$). We set $\mathbb{C}^{\mathrm{cv}}[[Q]] = \{f \in \mathbb{C}[[Q]] : \mathrm{CV}(f) \neq \emptyset\}$. Let $q \in \mathrm{CV}(f)$. There exist some constants $C_{f\rho}, \rho$ and $\check{\rho}$ such that $|q_1| < \check{\rho} < \rho, \ldots, |q_n| < \check{\rho} < \rho$ and $|f_{i_1,\ldots,i_n}|\rho^{i_1+\ldots+i_n} \leq C_{f\rho}$, for $i_1 \ldots, i_n \geq 0$. The* convergence module *of f at q is $(C_{f\rho}, \rho, \check{\rho})$.*

Recall $D_1^{j_1} \ldots D_n^{j_n} f$ is the *partial derivation* of order $j_1, \ldots, j_n \geq 0$ of f and is given by

$$\frac{D_1^{j_1} \ldots D_n^{j_n} f}{j_1! \ldots j_n!} = \sum_{i_1 \geq j_1, \ldots, i_n \geq j_n} f_{i_1,\ldots,i_n} \prod_{l=1}^{n} \binom{i_l}{j_l} q_n^{i_l - j_l}.$$

Definition 4. *The* polysystem *$\{A_i\}_{i=0,..,m}$ is defined by the Lie derivations $A_i = \sum_{j=1}^n A_i^j D_j$, where $A_i^j \in \mathbb{C}^{\mathrm{cv}}[[Q]]$. It is* linear *if there exist $\{M_i\}_{i=0,..,m} \in \mathcal{M}_{n,n}(\mathbb{C})$ s.t.*

$$A_i = \begin{pmatrix} q_1 & \ldots & q_n \end{pmatrix} M_i \begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix}.$$

Let $f \in \mathbb{C}^{\mathrm{cv}}[[Q]]$ and let $\{A_i\}_{i=0,1}$ be a polysystem. Let $(\rho, \check{\rho}, C_f)$ and let $(\rho, \check{\rho}, C_i)$, for $i = 0, 1$, be convergence modules of f and $\{A_i^j\}_{j=1,..,n}$ respectively at $q \in \mathrm{CV}(f) \Cap_{i=0,1,j=1,..,n} \mathrm{CV}(A_i^j)$. We denote by $(A_i f)_{|_q}$ the evaluation at q of $A_i f$. Let us consider the system

$$y(z) = f(q(z)), \quad \text{where} \quad dq(z) = A_0(q)\omega_0(z) + A_1(q)\omega_1(z). \tag{43}$$

Let us consider then the following generating series, $\sigma f_{|_{q(z_0)}}$ and the Chen series $S_{z_0\rightsquigarrow z}$

$$\sigma f_{|_{q(z_0)}} = \sum_{w\in X^*} A_w f_{|_{q(z_0)}} w \quad\text{and}\quad S_{z_0\rightsquigarrow z} = \sum_{w\in X^*} \alpha_{z_0}^z(w)w, \tag{44}$$

where $A_w = \mathrm{Id}, \alpha_{z_0}^z(w) = 1$ if $w = \epsilon$ and $A_w = A_v A_i, \alpha_{z_0}^z(w) = \int_{z_0\rightsquigarrow z} \alpha_{z_0}^t(v)\omega_i(t)$ if $w = vx_i$.

Since $S_{z_0\rightsquigarrow z}$ and $\mathrm{L}(z)\mathrm{L}(z_0)^{-1}$ satisfy (22) taking the same value at z_0 then $S_{z_0\rightsquigarrow z} = \mathrm{L}(z)\mathrm{L}(z_0)^{-1}$. Hence, the asymptotic behaviour of L in (22) gives [6]

$$S_{\varepsilon\rightsquigarrow 1-\varepsilon} \sim e^{-x_1\log\varepsilon} Z e^{-x_0\log\varepsilon} \quad\text{for}\quad \varepsilon\to 0^+, \tag{45}$$

and the output y of (43) is given by $y(z) = \langle \sigma f_{|_{q(z_0)}} || S_{z_0\rightsquigarrow z}\rangle = \sum_{w\in X^*} A_w f_{|_{q(z_0)}} \alpha_{z_0}^z(w)$ [14].

Let $\eta = q(z_0)$ and suppose that $f(q) = \lambda q$ with $\lambda \in \mathcal{M}_{1,n}(\mathbb{C})$. If $\{A_i\}_{i=0,1}$ is linear, then, by Definition 4, let $M_i = \mu(x_i)$, for $i = 0, 1$. Thus, $\sigma f_{|_{q(z_0)}} = \sum_{w\in X^*}[\lambda\mu(w)\eta]w$ is a rational power series of representation (λ, μ, η) and it is a generating series of the differential system of rank n, or equivalently of the linear differential equation of order n with singularities in $\{0, 1, \infty\}$.

Example 7 (hypergeometric equation).

$$z(1-z)\ddot{y}(z) + [t_2 - (t_0 + t_1 + 1)z]\dot{y}(z) - t_0t_1y(z) = 0.$$

Let $q_1(z) = y(z)$ *and* $q_2(z) = z(1-z)\dot{y}(z)$. *One has*

$$\begin{pmatrix} dq_1 \\ dq_2 \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ -t_0t_1 & -t_2 \end{pmatrix}\omega_0 - \begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}\omega_1\right]\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

Here $y = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$, $M_0 = -\begin{pmatrix} 0 & 0 \\ t_0t_1 & t_2 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}$ *and* $\eta = \begin{pmatrix} q_1(z_0) \\ q_2(z_0) \end{pmatrix}$.

Thus, the solution of these equations can be obtained by examining the linear representation of generating series. The Drinfel'd equation allows to study the asymptotic behaviour, the functional equations and to compute the mondromy groups, the Galois differential groups.

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